QMS 110: Exponential Functions

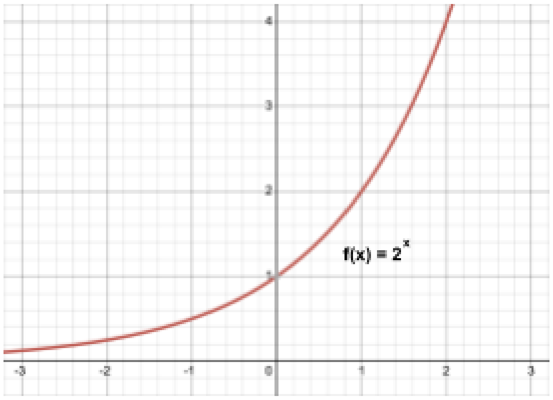
TRAIN TO LEARN EFFECTIVELY: TIP SHEETS

CONTENT BY: LAUREN RODRIGUEZ

WHAT ARE EXPONENTIAL FUNCTIONS?

Exponential functions are a type of function where the variable is an exponent.

$f(x) = a^x$, where $a > 0$, and $a \neq 1$



CHARACTERISTICS OF EXPONENTIAL FUNCTIONS

	When $a > 1$	When $0 < a < 1$
domain	$(-\infty, \infty)$	$(-\infty, \infty)$
range	$(0, \infty)$	$(0, \infty)$
x-intercept	none	none
y-intercept	$(0, 1)$	$(0, 1)$
contains	$(1, a), (-1, \frac{1}{a})$	$(1, a), (-1, \frac{1}{a})$
asymptote	$x - axis$, line $y = 0$	$x - axis$, line $y = 0$
basic shape	increasing	decreasing

EXPONENT LAWS

Product rules	$a^x a^y = a^{x+y}$
	$a^x b^x = (ab)^x$
Quotient rules	$\frac{a^x}{a^y} = a^{x-y}$
	$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$
Power rules	$(a^x)^y = a^{xy}$
	$a^{y^x} = a^{(y^x)}$
	$\sqrt[y]{a^x} = a^{\frac{x}{y}}$
	$a^{\frac{1}{x}} = \sqrt[x]{a}$
Negative exponents	$a^{-x} = \frac{1}{a^x}$

EXPONENTIAL FUNCTION PROPERTIES

$a > 1$ and $a \neq 1$, if $a^x = a^y$, then $x = y$

Example:

Solve for x: $3^x = 81$ ➡ $3^x = 3^4$ ➡ $x = 4$

EULER'S NUMBER

Euler's number, represented by e, equals to about 2.718. More notably, e is the base of the inverse natural logarithm function, $f(x) = e^x$. e is also an important element in commonly used exponential function formulas.

COMMON EXPONENTIAL FUNCTION FORMULAS

Compound Interest: The amount of interest accumulated from both the initial deposit amount, and the accrued interest from prior periods.

Formula: $A = P(1 + \frac{r}{m})^{mt}$, where

A = amount/future value after t years
P = principal/present value
r = annual rate
m = number of compounding periods a year
t = time (in years)

Exponential Growth: The process of a starting amount increasing by a rate that also grows proportionally to the size of the amount. (*This definition encompasses compound interest, however, this formula is used more generally, typically for topics such as population and bacteria growth, the spread of a virus, etc.*)

Formula: $A = A_0 e^{rt}$, where

A = total amount after time t
A_0 = starting amount
r = rate of growth
t = time

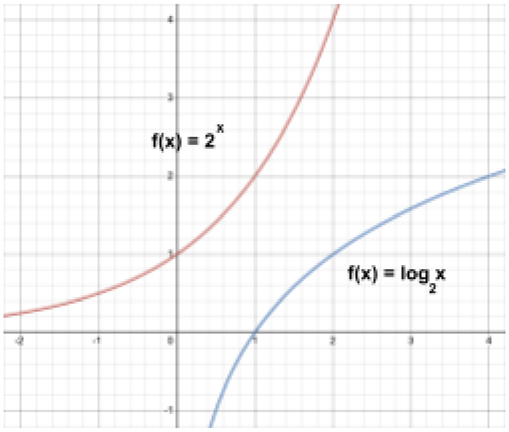
Exponential Decay: The process of a starting amount decreasing by a rate that also decays proportionally to the size of the amount. (*Similar to exponential decay, this formula is used more generally, typically for topics such as depreciation, decay of a substance, temperature cooling, departure of a population in a city over time, etc.*)

Formula: $A = A_0 e^{-kt}$, where

A = total amount after time t
A_0 = starting amount
k = decay rate
t = time

HOW DO EXPONENTIAL FUNCTIONS CONNECT TO LOGARITHMIC FUNCTIONS?

Exponential functions are the inverse of logarithmic functions. This means if you take the base function $f(x) = a^x$, if you invert the variables, you will get the the equation $x = a^y$ (as $f(x) = y$). This equation is equivalent to base logarithmic function $f(x) = log_a x$ (*through the laws and properties of logarithms*).



For more on logarithms, check out our tip sheet on Introduction to Logarithms.

TIPS ON SOLVING EXPONENTIAL FUNCTION QUESTIONS

Exponential function questions typically involve either an expression where you are required to expand and simplify (to its simplest form or into a singular term), or an equation where you are required to solve for the variable.

Otherwise, exponential function questions come in word problems that are typically associated with its own formula (i.e. compound interest, exponential growth or decay). These questions involve applying the right formula and solving.

Oftentimes there is a lot of crossover between exponential and logarithmic functions when solving questions (*since they are inverse of each other*), as you are likely using the rules, laws, and properties from both functions when solving.

When simplifying/solving these questions, consider the following:

- **Focus on one section of the expression instead of the entire thing.**
  - It can be overwhelming to view the whole expression all at once. Instead, focus on one part, and see what you can do or simplify within that part before branching outward.
  - For example, if you have a fraction, focus on simplifying only the numerator first. If you have an expression with multiple terms, focus on only one of them.
- **Have a reference to all the exponential (and logarithmic) rules, laws, and properties on hand, and see what parts of your expression match.**
  - Have a one page summary (or even better, your crib sheet) with all the possible rules next to you when practicing. It helps you determine what rule to apply.

- See if you can pick out any part of an expression that matches the formatting or any of the rules. Ask yourself, “does what the rule equates get you one step closer to fully simplifying or solving the equation?”
- The more questions you practice, the faster you will get at identifying the appropriate rules to use.

- **Be aware that cross over is highly likely.**
  - Whether you are starting out with an exponential or logarithmic expression/equation, you are very likely to be needing the rules from both in your work. Do not be afraid of switching between the two forms (*for example, with the conversion formula*).

EXPONENTIAL FUNCTION QUESTION EXAMPLE

Simplify the following expression: $\frac{(3x^2y^3)^2 \bullet (2x\sqrt{y^4})}{6x^3y}$.

TIP: For demonstrative purposes, we will go step-by-step for simplifying this expression. As you practice simplifying and solving exponential function expressions/equations, it is possible (and highly encouraged) for you to work on multiple areas of an expression within the same step (e.g. working on simplifying parts of the numerator and denominator simultaneously).

With a very complex expression, we will first start narrowing our focus and simplifying this expression part by part.

First, we will look at the first term in the expression [$(3x^2y^3)^2$] and seeing what exponent rules can be applied. Looking closely, his term matches with the power rule: $(a^x)^y = a^{xy}$. Using this rule, we can multiply the outward most exponent (2) to the inner exponents on the 3 (*as a constant number has a default variable of 1*), x, and y variables. This is what the term will look like:

$$(3x^2y^3)^2 \;\Rightarrow\; 3^{1\bullet 2}x^{2\bullet 2}y^{3\bullet 2} \;\Rightarrow\; 3^2x^4y^6 \;\Rightarrow\; 9x^4y^6$$

There is no further simplifying we can do to this term.

Next, we will simplify the other term in the numerator [$(2x\sqrt{y^4})$]. Looking at the exponent rules and what we can simplify, the y variable matches another power rule that we can use: $\sqrt[y]{a^x} = a^{\frac{x}{y}}$. Knowing that the default root number for a square root is 2, we can use this power rule and simplify. This is what the term will look like:

$(2x\sqrt{y^4}) \rightarrow 2xy^{\frac{4}{2}} \rightarrow 2xy^2$

There is no further simplifying we can do to this term.

Now looking at the entire numerator, we have the following: $9x^4y^6 \bullet 2xy^2$. We can simplify this expression by multiplying the two terms together, keeping in mind the product rule: $a^x a^y = a^{x+y}$. This is what the term will look like:

$9x^4y^6 \bullet 2xy^2 \rightarrow (9 \bullet 2)(x^4 \bullet x^1)(y^6 \bullet y^2) \rightarrow 18(x^{4+1})(y^{6+2}) \rightarrow 18x^5y^8$

There is no further simplifying we can do to this term.

Now, turning to the denominator, there is no further simplifying we can do the term there. Instead, we can focus on the entire expression and see how we can further simplify from there: $\frac{18x^5y^8}{6x^3y}$. The term in its current state can be simplified using the following quotient rule: $\frac{a^x}{a^y} = a^{x-y}$. This is what the expression will look like:

$\frac{18x^5y^8}{6x^3y} \rightarrow \frac{18}{6} \bullet \frac{x^5}{x^3} \bullet \frac{y^8}{y^1} \rightarrow 3x^{5-3}y^{8-1} \rightarrow 3x^2y^7$

There is no further simplifying we can do to this expression. Therefore, the final, simplified expression is $3x^2y^7$.

EXPONENTIAL FUNCTION QUESTION EXAMPLE

Solve for x. $10^x = 2$ (refer to [Introduction to Logarithms] and to logarithm function rules for reference).

Looking at all the exponential function rules, none of them apply. The closest one to the format of this equation is the exponential function property: if $a^x = a^y$, then $x = y$. In this case, there is no easy way to have the bases (10 and 2) to be the same so we can apply the property. From here, we can either take the log on both sides (*based on the idea that we can multiply an equation by anything, as long as we do so to every element so as to not change the value*), or use the conversion formula. We will use the conversion formula in this example.

Conversion formula: $B^p = N$ ➡ $p = log_B N$

$10^x = 2$ ➡ $x = log_{10} 2$

In this form, the equation matches the change of base logarithmic rule: $log_B N = \frac{logN}{logB}$.

Now we can apply the rule to our example: $x = log_{10} 2$ ➡ $x = \frac{log2}{log10}$.

Based on the characteristic of logarithms, whenever there is an expression of log on its own with no subscript, the default number is 10 (i.e. log_{10}). Knowing this, when looking at the denominator of this expression, we can see another logarithmic property can be applied: $log_a a = 1$.

So, we can apply this property to our example: $x = \frac{log2}{log10}$ ➡ $x = \frac{log2}{1}$ ➡ $x = log2$

This is the most simplified form. Therefore, the value of x in this example is $log2$.