ITM 107 Tip sheet

MATRICES

ITM 107

Matrices

Definition:

- Data in a matrix is called an element or an entry
- Size: Number of rows x Number of columns

Type of matrices:

- **Vectors**:
  - Row matrix: 1 row, n columns ($n \in N$) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$
  - Column matrix: n rows, 1 column ($n \in N$) $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$
- **Square matrix:** Number of rows = Number of columns
- **Zero matrix:** all entries = 0
- **Identity matrix (I)**:
  - Square matrix
  - 1s down diagonally and 0s everywhere else
  - When multiplying a matrix by an identity matrix, the result is the same matrix
  - Example:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Equal matrices: 2 matrices are equal when:
  - They have the **same size**
  - Each entry of matrix A equals the corresponding entry of matrix B
- Transpose matrix: rows interchange with columns

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$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \\ 6 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 9 \end{bmatrix}$$

Operations:

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 0 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Sum:

- Both matrices must have the **same size** to perform the addition operation
- Sum entry by entry
- Example:

$$A+B = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \\ A_{31}+B_{31} & A_{32}+B_{32} \end{bmatrix} = \begin{bmatrix} 1+2 & 4+1 \\ 5+0 & 0+0 \\ 3+1 & 1+1 \end{bmatrix}$$

Difference:

- Both matrices must have the **same size** to perform the subtraction operation
- Subtract entry by entry
- Example:

$$A\text{-}B = \begin{bmatrix} A_{11}-B_{11} & A_{12}-B_{12} \\ A_{21}-B_{21} & A_{22}-B_{22} \\ A_{31}-B_{31} & A_{32}-B_{32} \end{bmatrix} = \begin{bmatrix} 1-2 & 4-1 \\ 5-0 & 0-0 \\ 3-1 & 1-1 \end{bmatrix}$$

Scalar multiplication:

$(c \in R)\ if\ c = 2$

- Multiply all entries by c

$$c.A = 2.A \begin{bmatrix} 2 \times A_{11} & 2 \times A_{12} \\ 2 \times A_{21} & 2 \times A_{22} \\ 2 \times A_{31} & 2 \times A_{32} \end{bmatrix} = \begin{bmatrix} 2 \times 1 & 2 \times 4 \\ 2 \times 5 & 2 \times 0 \\ 2 \times 3 & 2 \times 1 \end{bmatrix}$$

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Product:

- Number of columns of 1st matrix **MUST EQUAL** number of rows of 2nd matrix
- The product matrix will have the size:
  - Same number of rows as the 1st matrix
  - Same number of columns as the 2nd matrix
- Each entry in the product matrix = (Row of 1st matrix) x (Column of 2nd matrix), element by element, then sum all up
- Example:

$$C = \begin{bmatrix} 1 & 4 \\ 5 & 0 \\ 3 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

Matrix C's size is 3x2 (3 rows and 2 columns), and matrix D's size is 2x3 (2 rows and 3 columns). The product of matrix C and D will have the size of 3x3 (3 rows and 3 columns)

$$C \text{ x } D = \begin{bmatrix} C_{11} \times D_{11} + C_{12} \times D_{21} & C_{11} \times D_{12} + C_{12} \times D_{22} & C_{11} \times D_{13} + C_{12} \times D_{23} \\ C_{21} \times D_{11} + C_{22} \times D_{21} & C_{21} \times D_{12} + C_{22} \times D_{22} & C_{11} \times D_{13} + C_{12} \times D_{23} \\ C_{31} \times D_{11} + C_{32} \times D_{21} & C_{31} \times D_{12} + C_{32} \times D_{22} & C_{31} \times D_{13} + C_{32} \times D_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + 4 \times 0 & 1 \times 3 + 4 \times 4 & 1 \times 1 + 4 \times 0 \\ 5 \times 2 + 0 \times 0 & 5 \times 3 + 0 \times 4 & 5 \times 1 + 0 \times 0 \\ 3 \times 2 + 1 \times 0 & 3 \times 3 + 1 \times 4 & 3 \times 1 + 1 \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 19 & 1 \\ 10 & 15 & 5 \\ 6 & 13 & 3 \end{bmatrix}$$

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Solving a system of linear equations:

Gauss-Jordan Elimination:

This method is used to simplify the augmented matrix until the left-hand side is in reduced form, making the solution easier to read. Depending on the problem, you can apply any of the row operation steps as needed. These steps are guidelines; you can repeat or skip any of them as necessary to reach the reduced form matrix.

- Interchange 2 rows
- Multiply or divide a row by a non-zero constant
- Add a multiple of one row to another row

Reduced form matrix:

- **Leading 1s:** Each nonzero **row** has a leading 1 (first non-zero entry in the row)
- **Stair-step pattern:** each leading 1 is to the right of the leading 1 in the row above it
- **Zeros above and below pivots:** Each **column** contains a **leading 1,** and has zero everywhere in that column
- An identity matrix is a reduced form matrix. However, **not all reduced form matrices are an identity matrix**.

When a matrix is in its reduced form:

- If it is an identity matrix, then there is a unique solution
- Example:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

When we put the augmented matrix back in the form of a system of linear equations, this result shows:

$$\begin{cases} x = 1 \\ y = 5 \\ z = 3 \end{cases}$$

- When the solution is **not an identity matrix**:
  - If the bottom row has all 0s, then it is a **non-unique solution**
  - Example:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

    When we put the augmented matrix back in the form of a system of linear equations, this result shows:

$$\begin{cases} x + 2z = 1 \\ y + z = 5 \\ 0 = 0 \end{cases} \rightarrow \begin{cases} x = 1 - 2x \\ y = 5 - z \\ z \in \mathbb{R} \end{cases}$$

    In this case, z is an independent variable. For any selected z, there is a corresponding x and y

  - If a row has all 0s on the left-hand side and a number on the right-hand side, then there is **no solution**
  - Example:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

    When we put the augmented matrix back in the form of a system of linear equations, this result shows:

$$\begin{cases} x + 2z = 1 \\ y = 5 \\ 0 = 1 \end{cases}$$

    Since the bottom equation is logically impossible, we say there is no solution.

Inverse matrix:

To solve a system of linear equations: $AX = B \rightarrow A^{-1}(AX) = A^{-1}B \rightarrow X = A^{-1}B$

2 ways to find the inverse matrix:

- Use formula **(ONLY APPLY FOR 2X2 MATRIX)**

  Given A is a 2x2 matrix, the inverse of A will be A^{-1}.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{Then } A^{-1} = \frac{1}{ad-bc} \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  Provided $ad - bc \neq 0$. If $ad - bc = 0$, then A^{-1} does not exist

- Use the identity matrix and row operations (can apply to **any size of square matrix**):
  - o Form an augmented matrix with the given matrix on the left-hand side and an identity matrix on the right-hand side. The identity matrix must be the same size as the given matrix
  - o Perform row operations to reach an identity matrix on the left-hand side; the right-hand side will be the inverse matrix
  - o Example:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \quad -R_1 + R_2 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & -1 & -1 & 1 \end{array}\right] \quad 2R_2 + R_1 \rightarrow R_1$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right] \quad -R_2 \rightarrow R_2$$

$$\text{The inverse of A is } A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Note: if A has no inverse, the reduction process on $\left[\begin{array}{c|c} A & I \end{array}\right]$ will yield a row of zeros in the left-hand side of the augmented matrix