QMS 130: Differentiation

RULES AND SPECIAL CASES

CONTENT BY: SARAH HYATOOLA

The Derivative

The derivative of a function f at a number a, denoted by $f'(a)$, is given by the following equation, provided that the limit exists:

$$f^{'}(a) = \frac{f(a+h)-f(a)}{h}$$

In general, the derivative of the function $f(x)$ at any given x is given by the following equation:

$$f^{'}(x) = \frac{f(x+h)-f(x)}{h}$$

For any value of x such that this limit exists, $f'(x)$ is referred to as the derivative of f.

Derivative Notation

Derivatives can be written using various notation styles, such as:

- $f'(x)$
- y'
- $\frac{dy}{dx}$ (Leibniz Notation)

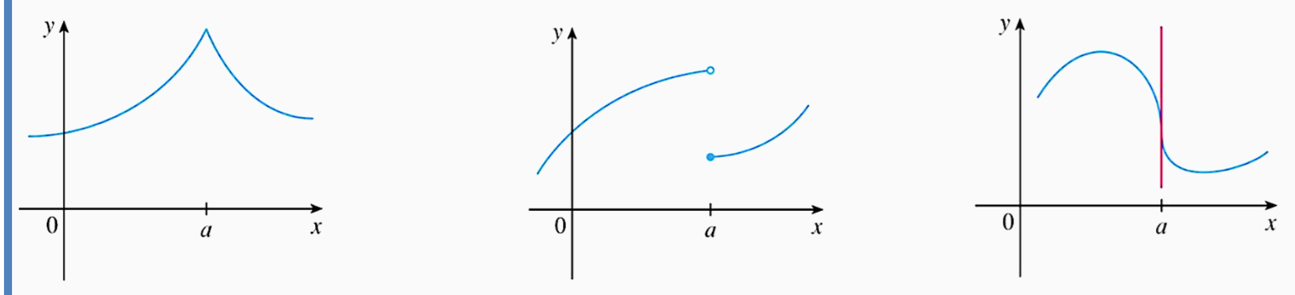
If we want to indicate the value of the derivative at a specific number a, then we use the following notation:

- $f'(a)$
- $\frac{dy}{dx}\Big|_{x=a}$

Differentiability

Functions fail to be differentiable in the following cases:

1. If the graph of a function has a ***corner*** or ***kink*** in it (no tangent on that point).
2. At any ***discontinuity points***.
3. If the curve has a ***vertical tangent line***.



(a) A corner

(b) A discontinuity

(c) A vertical tangent

Three ways for *f* not to be differentiable at *a*

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Rules of Differentiation

Name of Differentiation Rule	General Form of Differentiable Function(s)	Description of Differentiation Rule	General Form of Resulting Derivative
Constant Functions	$f(x) = c$ (given c is a real number)	The derivative of a constant function is zero.	$\frac{d}{dx}(c) = 0$
Power Rule	$f(x) = x^n$ (given n is a real number)	The derivative given by the power rule is obtained by: i) multiplying the term by ii) n (i.e., multiplying the term by the power of x); and iii) reducing the power of x to $n - 1$ (i.e., subtracting 1 from the power).	$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$
The Constant Multiple Rule	$cf(x)$ (given c is a constant and f is a differentiable function)	When differentiating a function f that has a constant multiple c, the constant multiple c is treated as a constant multiple of the derivative of f.	$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$

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The Sum and Difference Rules	$f(x) \pm g(x)$ (given f and g are both differentiable functions)	The derivative of the sum (or difference) of two functions, f and g, is exactly the same as the sum (or difference) of the derivative of f and the derivative of g. That is, the derivative of the sum (or difference) of two functions may be obtained by differentiating each function separately. Note that these rules extend to any number of functions.	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$
The Product Rule	$f(x)g(x)$ (given f and g are both differentiable functions)	The derivative of the product of two functions, f and g, is obtained by: i) multiplying f by the derivative of g (i.e., multiplying the first function by the derivative of the second function); then ii) multiplying g by the derivative of f (i.e., multiplying the second function by the derivative of the first function); and iii) adding the two terms.	$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$ <u>OR:</u> $(fg)' = f'g + fg'$

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The Quotient Rule	$\frac{f(x)}{g(x)}$ (given f and g are both differentiable functions)	The derivative of the quotient of two functions, where f is the numerator and g is the denominator, is obtained by: i) multiplying g by the derivative of f (i.e., multiplying the denominator by the derivative of the numerator); then ii) multiplying f by the derivative of g (i.e., multiplying the numerator by the derivative of the denominator); then iii) subtracting the term obtained in Step ii) from the term in Step i) [i.e., i) – ii)]; and iv) dividing the term obtained in Step iii) by the square of g (i.e., the square of the denominator).	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$ <u>OR:</u> $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

The Chain Rule	$F(x) = f(g(x))$ (given g is differentiable at x and f is differentiable at $g(x)$, such that the composite function F is differentiable at x)	The derivative of a composite function, $f(g(x))$ is obtained by: i) identifying $g(x)$ and assigning it to a variable, preferably u (i.e., stating "Let $u = g(x)$"); then ii) substituting u into $f(g(x))$ so that f is now expressed in terms of u [i.e., rewriting $f(g(x))$ as $f(u)$]; then iii) differentiating $f(u)$ with respect to u to obtain $f'(u)$; then iv) differentiating $g(x)$ with respect to x to obtain $g'(x)$; then v) multiplying the two terms obtained in Steps iii) and iv) [i.e., $f'(u) \bullet g'(x)$]; and vi) rewriting the term obtained in Step v) in terms of x [i.e., $f'(g(x)) \cdot g'(x)$].	$F'(x) = f'(g(x)) \cdot g'(x)$ <u>OR:</u> *<u>Leibniz Notation:</u>* Let $y = f(u)$, where $u = g(x)$. Then: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

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Additional Differentiation Topics

Derivatives of Logarithmic Functions

The logarithmic function $y = x$ is the inverse of the exponential function $y = b^x$. The exponential function is differentiable with respect to x, and as a result, it follows that the logarithmic function is also differentiable.

In general, the derivative of the logarithmic function is given by the following equation:

$$\frac{d}{dx}(x\ ) = \frac{1}{x \ln ln\ b}$$

When base $b = e$, the derivative of the logarithmic function is given by the following equation:

$$\frac{d}{dx}(\ln ln\ |x|\ ) = \frac{1}{x} \qquad (x \neq 0)$$

Implicit Differentiation

In general, most of the functions studied in QMS 130 are expressed ***explicitly*** as one variable. The most general example of this is $y = f(x)$. Other examples include:

- $y = x^2 + x - 1$
- $y = 2x^3 + \frac{1}{\sqrt{x}}$

However, some functions are defined ***implicitly*** by a relation between x and y. For example:

- $x^2 + y^2 = 4$ (Equation of a Circle)
- $x^3 + y^3 = 6xy$

In **implicit differentiation**, we differentiate both sides of the equation with respect to x and then we solve the resulting equation in terms of $\frac{dy}{dx}$ or y'.

For the purposes of QMS 130, we assume that given equations of y implicitly as a function of x are differentiable.