# Budgeted Steiner Networks: Three Terminals with Equal Path Weights

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## Abstract

Given a set of terminals in 2D/3D, the network with the shortest total length that connects all terminals is a Steiner tree. At the other extreme, with enough total length budget, every terminal can be connected to every other terminal via a straight line, yielding a complete graph over all terminals that connects every pair of terminals with a shortest path. In this work, we study a generalization of Steiner trees, asking what happens between these two extremes. For a given total length budget, we seek a network structure that minimizes the sum of the weighted distances between pairs of terminals. Focusing on three terminals with equal pairwise path weights, we characterize the full evolutionary pathway between the Steiner tree and the complete graph, which contains interesting intermediate structures.


## 1 Introduction

Consider a scenario in which three or more terminals (e.g., the black nodes $A, B$, and $C$ in Fig. 1) are to be connected using a (graph) network, the total length of which is limited.

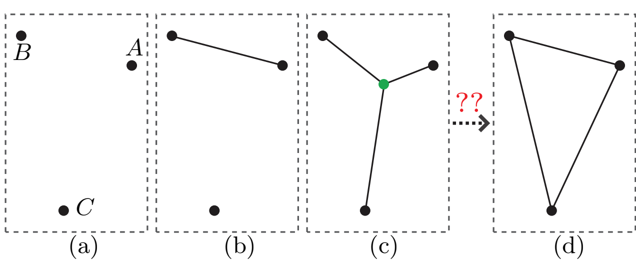


Figure 1: Evolution of a *budgeted Steiner network* over three (black) terminals as the budget increases. (a) Three terminals, $A, B$, and $C$, to be connected. (b) The minimal non-trivial network that connects two terminals. (c) The minimal network connecting all terminals, which is a Steiner tree. (d) With sufficient budget, the network is a complete graph. The question is, what happens between (c) and (d)?

At one extreme, the minimum length budget required to connect all terminals corresponds to the total length of the edges of a Steiner tree over the terminals (Fig. 1(c)). The well-known *Steiner tree problem* (`STP`) seeks optimal network structures for connecting a set of terminals while minimizing the total edge lengths [9,16]. `STP` generally asks for a minimally connected network, resulting in a topology that is a tree. At the other extreme, when there is no limit on the budget, the best network structure is clearly a complete graph over all terminals, where every pair of terminals are connected through a straight edge. Such a network ensures the shortest possible travel distance between any pair of terminals. What if, however, the budget falls between the two extremes?

To address the question, we propose the *budgeted Steiner network* (`BSN`) problem/model. As a natural generalization of `STP`, `BSN` seeks the best network structure for a given length budget to connect three or more terminals, which reside in $\mathbb{R}^d$ for some $d \geq 1$, such that the sum of the (weighted) distances between pairs of nodes are minimized. In this work, we mainly focus on the case of three terminals with $d = 2$ (for three terminals, $d = 2$ is the same as $d \geq 2$).

The generalization immediately leads to rich and interesting structures, even when only three terminals are involved. As the budget increases, the network structure changes continuously between a Steiner tree and a complete graph over the terminals, a few snapshots of which are illustrated in Fig. 2.

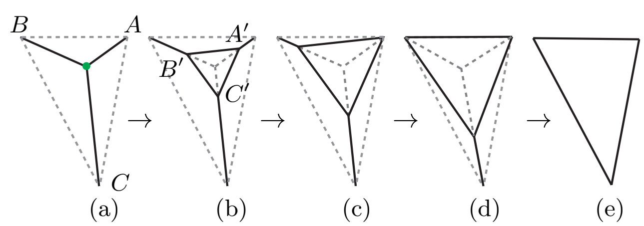


Figure 2: A spectrum of optimal Euclidean `BSN` network structures (solid lines) for three terminals in a typical setup, as the allowed budget increases.

As a summary of the full evolutionary pathway, if all internal angles of a $\triangle ABC$ are smaller than $2\pi/3$, the Steiner tree over terminals $A, B$, and $C$ has a Steiner point that is internal to the triangle (e.g., the green dot in Fig. 2). In this case, for a generic $\triangle ABC$ (that is, $\triangle ABC$ is not an isosceles triangle), as the budget increases past the length of the Steiner tree, an equilateral triangle $\triangle A'B'C'$ will "grow" out of the Steiner point

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(Fig. 2(b)) and continues to expand until one vertex of $\triangle A'B'C'$ meets one of the terminals, say $A$. Past this point, $\triangle A'B'C'$ continues to expand as an isosceles triangle with $A' = A$ fixed (Fig. 2(c)) as the budget continues to increase, until another vertex meets $B$ or $C$, say $B$. $\triangle A'B'C'$ then continue to expand with $A' = A$ and $B' = B$ fixed, and $C'$ moving toward $C$, until it fully coincides with $\triangle ABC$. If $\triangle ABC$ has one angle equal to or larger than $2\pi/3$, the evolutionary pathway is similar but shortened; the corresponding `BSN` does not have an initial phase containing an equilateral triangle.

The main contribution of this work is the rigorous characterization of the precise evolution pathway of a `BSN` as the available budget increases, for three arbitrarily located terminals. The analysis also implies an efficient algorithm for computing the optimal `BSN` structure for any given budget.

## 2 Related Work

`BSN` problems are closely related to `STP`s [8,9,16], which is a broad term covering a class of network optimization problems. An `STP` seeks a minimal network that connects a set of terminals (in Euclidean space or on graphs that are possibly edge/vertex weighted). There are four main cases: Euclidean, rectilinear, discrete/graph-theoretic [6,11], and phylogenetic [9]. Considering the paper's scope, we provide a brief literature review of Euclidean `STP`s.

The Euclidean `STP` asks the following question: given $n$ terminals in 2D or 3D, find a network that connects all $n$ points with the minimum total length (the discussion from now on will be limited to the 2D case). Obviously, the resulting network is a tree and may only have straight line segments; it may also require additional intermediary nodes to be added. These added nodes are called *Steiner points*. The study of Euclidean `STP` bears with it a long history; the initial mathematical study of the subject may be traced back to at least 1811 [3]. According to [12], key properties of Euclidean `STP` have been established in (as early as) the 1930s by Jarník and Kössler [10]. An interconnecting network $T$ is called a Steiner tree if it satisfies the following conditions [9]:

(a) $T$ is a tree,

(b) Any two edges of $T$ meet at an angle of at least $2\pi/3$, and

(c) Any Steiner point cannot be of degree 1 or 2.

These conditions turn out to be also relevant in our study of the `BSN` problem. The solution to an Euclidean `STP` must be a Steiner tree. Note that (b) implies a node of the network has a maximum degree of 3. Together, (b) and (c) imply that three edges must meet at a Steiner point forming angles of $2\pi/3$ in a pairwise manner (see Fig. 1). Because Euclidean Steiner trees assume minimal energy configurations, they also appear in nature. Indeed, it is possible to employ related natural phenomena (e.g., using rubber bands and soap film) to "compute" Euclidean Steiner trees [5,7,14].

Our study, which focuses on the case of three terminals with equal path weights, bears similarity with a recent study [4] which examines a related problem of characterizing the minimum dilation spanners on three terminals for a given budget. Whereas there exists a mild degree of similarity, we note that we independently developed our results, which provides an exact analysis of the full evolution pathway between the Steiner tree and the complete graph. On the other hand, the analytical result of [4] is mostly limited to the initial stage of the evolution.

Computing an Euclidean `STP` is NP-hard, although there is a polynomial time approximation scheme (PTAS) for solving it [2]. On the more practical side, fast methods including the GeoSteiner algorithm [15,17] have been developed building on the Melzak construction [13]. An open source implementation of the GeoSteiner algorithm is maintained [1].

## 3 Preliminaries

Let there be $n \geq 3$ *terminals* $N = \{v_1, \ldots, v_n\}$, distributed in some way in a $d$-dimensional unit cube, $d > 0$. For each pair of terminals $v_i$ and $v_j$, $1 \leq i < j \leq n$, let $w_{ij} \in (0, 1]$ denote the (relative) *weight* or *importance* of the route connecting $v_i$ to $v_j$. In practice, $w_{ij}$ may model the expected traffic flow from $v_i$ to $v_j$, for example. In an Euclidean *budgeted Steiner tree* (`BSN`) problem, straight line segments are to be added for connecting the $n$ terminals so that some or all of the terminals are connected. Similar to Steiner trees, intermediate nodes other than $v_1, \ldots, v_n$, which we call *anchors*, may be added. The terminals, anchors, and the straight line segments then form a graph containing one or more connected components. Under the constraint that the total length of the line segments does not exceed a *budget* $L$, the `BSN` problem seeks a network structure that minimizes the objective

$$J(L) = \sum_{1 \leq i < j \leq n} w_{ij} d_{ij}, \tag{1}$$

in which $d_{ij}$ denotes the shortest distance between $v_i$ and $v_j$ on the network. If no path exists between $v_i$ and $v_j$, let $d_{ij}$ be some very large number.

In the current work, we examine the case of $n = 3$ and $w_{ij} = 1$ for all $1 \leq i, j \leq 3$, $i \neq j$, i.e., paths between pairs of terminals are equally important. Let the three terminals be $A, B$, and $C$, we are looking for a `BSN` minimizing the sum $d_{AB} + d_{BC} + d_{AC}$ subject to the budget $L$. For a fixed $L$, let $N(L)$ denote the optimal

BSN structure. Let $L_{\texttt{ST}}$ be the budget $L$ when $N(L)$ is a Steiner tree. For convenience, let $N_{\texttt{ST}} := N(L_{\texttt{ST}})$.

## 4 Anchor Structures and Steiner Triangles

### 4.1 Basic Properties of Anchors

First, we note each anchor must have degree three.

**Lemma 1 (Degree of Anchors)** *For three terminals, any anchor must have degree exactly three.*

**Proof.** Each anchor must connect at least three line segments; otherwise, the anchor point and the involved line segments only cause increases to the objective $d_{AB} + d_{BC} + d_{AC}$. An anchor's degree also cannot be four or larger when there are only three terminals, because each outgoing edge from an anchor must be on a shortest path to a unique terminal, if we are to minimize Eq. (1). But there are only three terminals. □

We analyze what happens when $L = L_{\texttt{ST}} + \varepsilon$ for small $\varepsilon > 0$, for the case where the Steiner point lies inside $\triangle ABC$, which happens when all angles of $\triangle ABC$ are smaller than $2\pi/3$. Due to continuity, the resulting structure that minimizes Eq. (1) must be a perturbation of $N_{\texttt{ST}}$ (e.g., Fig. 1(b)). This means that $N(L_{\texttt{ST}}+\varepsilon)$ must start "growing" at the Steiner point. We want to understand how $N(L_{\texttt{ST}}+\varepsilon)$ evolves for small $\varepsilon$. This raises the following questions: (1) how many line segments are in $N(L = L_{\texttt{ST}} + \varepsilon)$ and (2) how do they come together? We note that $N(L_{\texttt{ST}} + \varepsilon)$ must contain more than three straight line segments. Otherwise, $N(L_{\texttt{ST}} + \varepsilon)$ will still be a tree but with $d_{AB}+d_{BC}+d_{AC} = 2(L_{\texttt{ST}}+\varepsilon) > 2L_{\texttt{ST}}$, i.e., $J(L) > J(L_{\texttt{ST}})$.

To answer above-mentioned questions, we start with establishing essential properties of anchors, concerning their locations, degrees, and numbers. It is clear that anchors must always fall within $\triangle ABC$; otherwise, an outside anchor (on the convex hull of all terminals and anchors) can be "retracted" toward the boundary of $\triangle ABC$ to reduce both the budget and the objective function value. In fact, anchors cannot reside on the boundary of $\triangle ABC$, as shown in the following lemma.

**Lemma 2 (Interiority of Anchors)** *For terminals $A, B$, and $C$, any anchor must fall in the interior of $\triangle ABC$, excluding its perimeter.*

**Proof.** Consider the setting illustrated in Fig. 3 where only a portion of $\triangle ABC$ is drawn. Suppose that $D$ is the only anchor on $AC$ and the horizontal line segment passing through $D$ and $D'$ is part of an optimal network structure. For the setup, $DD'$ must be part of the shortest path on the optimal network that connects $A$ to $B$ as well as $C$ to $B$; the entire $AC$ must also be part of the network that connects $A$ and $C$.

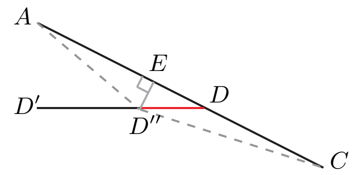


Figure 3: Moving $C'$ along $C'C$ for a small amount.

We claim that such a configuration cannot be optimal. To see this, retract $D$ along $DD'$ by some small distance of $|DD''|$. This reduces the budget by $\Delta L = |DD''|+(|AC|-|AD''|-|CD''|)$. At the same time, the cost reduction is $\Delta J = 2|DD''|+(|AC|-|AD''|-|CD''|)$.

Let $E \in AC$ be a point such that $D''E \perp AC$. It is straightforward to derive that $|ED''| \gg |CD''|-|CE|$ and $|ED''| \gg |AD''|-|AE|$ for sufficiently small $|ED''| > 0$. Therefore, $|DD''| \geq |ED''| > (|AD''|+|CD''|-|AC|)$. This means that for small $|DD''|$, both $\Delta L$ and $\Delta J$ are positive, i.e., we can reduce budget and at the same time reduce the cost by retracting $D$ along $DD'$ to $D''$. This means that $D$ cannot be an anchor. If $D$ is not the only anchor on $AC$, the same proof works assuming $D$ is the lowest anchor. □

Building on Lemmas 1 and 2, we show that there can be at most three anchors for three terminals.

**Lemma 3 (Number of Anchors in $N(L_{\texttt{ST}} + \varepsilon)$)** *When all angles of $\triangle ABC$ are below $2\pi/3$, for small $\varepsilon > 0$, $N(L_{\texttt{ST}} + \varepsilon)$ contains three anchors that forms a triangle inside $\triangle ABC$.*

**Proof.** By Lemma 1, all anchors have degree three. If there is only a single anchor that is not the Steiner point, then $N(L_{\texttt{ST}} + \varepsilon)$ still has a tree structure. This tree is different from $N_{\texttt{ST}}$ which is minimal, so the new tree must have a larger objective function value which cannot be optimal.

If there are two anchors, each with degree three, then both of them cannot be connected to all of $A, B$, and $C$; there must be exactly five line segments in $N(L_{\texttt{ST}} + \varepsilon)$, one of which connects the two anchors. This leaves four line segments connected to the three terminals, which means that two of these line segments must reach the same terminal. This will induce a total budget that cannot be an arbitrarily small amount above $L_{\texttt{ST}}$ when the Steiner point is inside $\triangle ABC$. That is, this is impossible with a budget $L_{\texttt{ST}} + \varepsilon$ for small $\varepsilon > 0$.

There cannot be more than three anchors when there are only three terminals. To establish this, we note that a shortest path between any two terminals, when there are three terminals in total, can make at most two "turns" due to path sharing. To see this, consider the shortest path $P_{AB}$ between terminals $A$ and $B$. $P_{AB}$ may bend at most two times, once to share with a path from $A$ to $C$ and once to share with a path from $B$ to $C$. If $P_{AB}$ bends once, say at an anchor $A'$, then both $AA'$ or $A'B$ must be on a shortest path to $B$ and we must have a tree. This is not possible under the assumption that $\varepsilon$ is small, so there can only be one edge coming out of a terminal. Therefore, each shortest path between two terminals must bend exactly twice at two anchors. The three shortest paths then have a total of six anchors. Because each anchor is shared by two shortest paths, there can only be three unique anchors that form a triangle. □

### 4.2 Steiner Triangle for Three Anchors

Having shown that there are three anchors, let the anchor closest to $A, B$ and $C$ be $A', B'$, and $C'$, respectively. This suggest that $N(L_{\texttt{ST}} + \varepsilon)$ contains six line segments $AA'$, $BB'$, $CC'$, $A'B'$, $A'C'$, and $B'C'$. We call $\triangle A'B'C'$ that "grows" out of the Steiner point a *Steiner triangle*. Next, we establish that $\triangle A'B'C'$ is an equilateral triangle, starting with showing that its three internal angles are bisected by $AA'$, $BB'$ and $CC'$. The objective Eq. (1), $d_{AB} + d_{BC} + d_{AC}$ for the current setting, translates to

$$\begin{aligned} J(L_{\texttt{ST}} + \varepsilon) = & 2|AA'| + 2|BB'| + 2|CC'| \\ & + |A'B'| + |A'C'| + |B'C'|. \end{aligned} \tag{2}$$

**Lemma 4 (Bisector of Steiner Triangle)** *For terminals $A, B$, and $C$ with a Steiner point, let $N(L_{ST} + \varepsilon)$ be composed of the Steiner triangle $\triangle A'B'C'$ and segments $AA'$, $BB'$ and $CC'$. Then an angle of $\triangle A'B'C'$ is bisected by the line passing the corresponding anchor and the terminal the anchor is connected to.*

**Proof.** See the Appendix for the technical proof based on infinitesimal analysis. □

Before moving on to showing that $\triangle A'B'C'$ is equilateral, we note that Lemma 4 does not depend on $\varepsilon$ being small. Moreover, the result continues to hold if there are one or two anchors, which can be readily verified.

**Lemma 5 (Anchor Bisector)** *For terminals $A, B$, and $C$, suppose $C'$ is an internal anchor connected to $C$ in an optimal network structure $N(L)$. Then $CC'$ bisects the angle formed by the other two outgoing edges from $C'$.*

We now prove a key structural property of `BSN` for three terminals involving three anchors.

**Theorem 6 (Steiner Triangle for Three Anchors)** *For terminals $A, B$, and $C$ with a Steiner point, assume that $N(L_{ST} + \varepsilon)$ is composed of the Steiner triangle $\triangle A'B'C'$ and segments $AA'$, $BB'$ and $CC'$. Then $\triangle A'B'C'$ is equilateral with its center being the Steiner point of the terminals. The center of $\triangle A'B'C'$ is the intersection point of $AA', BB'$ and $CC'$.*

**Proof.** See the Appendix. □

From Theorem 6, we can draw the following conclusion. For three terminals with a Steiner point, as the budget $L$ goes just beyond $L_{\texttt{ST}}$, an equilateral triangle will "grow" out the Steiner point toward the terminals. Moreover, whenever there are three anchors, they must form an equilateral triangle. All such equilateral triangles have their vertices lying on the line segments formed by the terminals and the Steiner point, as illustrated in Fig. 4. We have not yet show, however, that as $L$ grows, the anchors cannot go from three to fewer and then become three again. We delay this after the structures with fewer anchors are characterized.

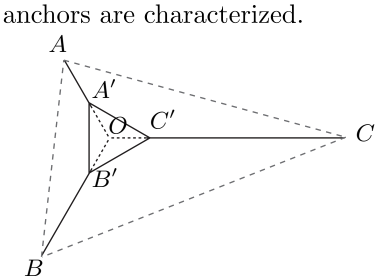


Figure 4: For three terminals with a Steiner point (which is always internal), when there are three anchors, they always form an equilateral triangle.

### 4.3 One and Two Anchors

If there are two anchors, they must both be connected to one shared terminal, say $A$, and each connecting to a unique terminal in $B$ and $C$. Let the anchors be $B'$ and $C'$. $N(L)$ then consists of five segments $AB'$, $AC'$, $BB'$, $CC'$, and $B'C'$. It can be shown that $\triangle AB'C'$ is an isosceles triangle (see, e.g., Fig. 5).

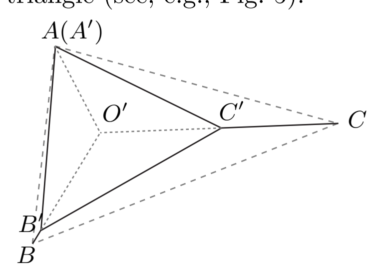


Figure 5: For three terminals with a Steiner point, when there are two anchors, they always form an isosceles triangle with one of the terminals.

**Proposition 1 (Steiner Triangle for Two Anchors)** *For terminals $A, B$, and $C$ with a Steiner point, if the optimal network $N(L)$ has two anchors $B', C'$, then these two anchors form an isosceles triangle with one of the terminals, e.g., $A$. $AB' = AC'$.*

**Proof.** See the Appendix. □

Following the same line of reasoning, when there is a single anchor in an optimal network $N(L)$, e.g., $C'$ that is connected to $A, B$, and $C$, if $C'$ is not the Steiner point, $N(L)$ must contain one of $AB$, $BC$, and $AC$. Suppose $N(L)$ contains $AB$, then all we know is that $CC'$ must bisect $\angle AC'B$. See Fig. 2(d) for an example.

## 5 Evolution of the Budgeted Steiner Network

### 5.1 With Steiner Point

Having established the optimal configuration when there are 1-3 anchors, we now piece them together to understand the evolution of the network. Intuitively, as the budget $L$ increases, the evolution of the optimal network $N(L)$ would look like that shown in Fig. 2, going from Steiner tree to having three anchors, then two, then one, and finally becoming the triangle of the three terminals. To show this is the actual network evolution pathway, however, we must show that there cannot be discrete jumps in `BSN` structures, e.g., going from three anchors to two anchors and then back to three anchors.

We proceed to show that the sequence in Fig. 2 is indeed how $N(L)$ evolves as $L$ increases by analyzing how $J(L)$ changes as $L$ changes, i.e., $\frac{dJ}{dL}$.

**Lemma 7 (Rate of Change at Anchors)** *For terminals A, B, and C, let $C'$ be an anchor connected to C. Let the angle formed by the other two edges emanating from $C'$ other than $CC'$ be $2\alpha$. As $C'$ moves closer to C, the rate of change to the objective function $\frac{dJ}{dL}$ due to the change to $CC'$ is*

$$\frac{dJ}{dL} = \frac{2\cos\alpha - 2}{2\cos\alpha - 1}. \tag{3}$$

**Proof.** Fig. 6 shows the setting where $C'$ is moved along $C'C$ for a small amount. By the bisector Lemma 5, the addition of length (in green) to the two edges coming out of $C'$ that are not $CC'$ is $2|EC'|$ while the reduction of length to $|CC'|$ is $|C'E|/\cos\alpha$ (the red segment). Therefore, the change to the budget due to this is $\Delta L = 2|C'E| - |C'E|/\cos\alpha$.

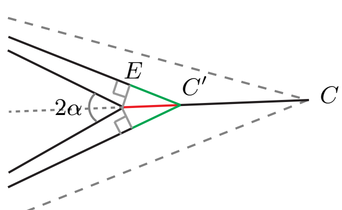


Figure 6: Moving $C'$ along $C'C$ for a small amount.

On the other hand, the change to the objective function value is $\Delta J = -(2|C'E|/\cos\alpha - 2|C'E|)$ because $C'C$ contributes to two shortest paths. Dividing $\Delta J$ over $\Delta L$ yields Eq. 3. □

**Proposition 1 (Range of Change, Three Anchors)** *For three terminals, when there are three anchors,*

$$\frac{dJ}{dL} = \frac{1-\sqrt{3}}{2}. \tag{4}$$

**Proof.** For three anchors, $\alpha$ in Eq. (3) is $\pi/6$. We then have $dJ/dL = (\sqrt{3}-2)/(\sqrt{3}-1) = (1-\sqrt{3})/2$. □

**Proposition 2 (Range of Change, 1-2 Anchors)** *For three terminals, when there are one of two anchors, let the angle formed at the anchor belonging to the triangle structure of the network be $2\alpha$, then,*

$$\frac{dJ}{dL} = \frac{2\cos\alpha - 2}{2\cos\alpha - 1}. \tag{5}$$

Since $0 < 2\alpha \leq \pi/2$, $\alpha \in (0, \pi/4]$. Let $\cos\alpha = x$, $x \in [\frac{\sqrt{2}}{2}, 1)$. Eq. 3 becomes $g(x) = \frac{2x-2}{2x-1}$. It is straightforward to derive (using derivatives) that $g(x)$ is negative on the given range of $x$ and monotonically increases to 0 as $x \to 1$. This means, with reference to Fig. 6, that the magnitude of $\frac{dJ}{dL}$ becomes smaller as $C'$ gets closer to $C$ ($\alpha$ decreases). This allows us to show that $J(L)$ decreases faster when there are more anchors. We begin with showing that internal angles at anchors cannot exceed $\pi/3$.

**Lemma 8 (Feasible Anchor Configurations)** *For three terminals and an optimal Steiner network, the internal angles of the triangular structure of the network at non-terminal anchors are always no more than $\pi/3$.*

**Proof.** For three anchors, we have shown they must assume an equilateral triangle configuration. Suppose that in a two-anchor network configuration, the optimal network has internal angles at non-terminals anchors larger than $\pi/3$. For example, suppose that in Fig. 5, $\angle AB'C' = \angle AC'B' > \pi/3$. This requires that $\angle B'AC' < \pi/3$. Now, suppose we push down the triangle $A'B'C'$ along $AA'$ by a small $\delta > 0$ and retract along $B'B$ and $C'C$ so that $L$ remains unchanged. Because $0 > \frac{dJ}{dL}|_{A'} > \frac{dJ}{dL}|_{B'} = \frac{dJ}{dL}|_{C'}$, this means that $J$ will actually decrease due to the change. Therefore, the configuration cannot be optimal.

The same argument also applies to the single anchor case: if the internal angle at the single anchor is larger than $\pi/3$, the at least one of the two other internal angles must be smaller than $\pi/3$. □

We are now ready to establish the evolution pathway of the optimal Steiner network for three terminals with Steiner points.

**Theorem 9 (BSN Evolution, with Steiner Point)** *For terminals $A, B$, and $C$ with a Steiner point $O$, as the budget $L > L_{ST}$ increases, the optimal Steiner network $N(L)$ will first grow an equilateral triangle, $\triangle A'B'C'$, out of $O$ toward the three terminals. The internal angles of $\triangle A'B'C'$ are bisected by $AA', BB'$ and $CC'$. The growth continues until one of the anchors, say $A'$, reaches terminal $A$, corresponding to the largest internal angle of $\triangle ABC$. Then, an isosceles triangle continuous to grow in place of the equilateral triangle, with its two internal angles $\angle AB'C'$ and $AC'B'$ bisected by $BB'$ and $CC'$, respectively, until one of the two anchors $B'$ reaches a second terminal, say $B$, that corresponds to the second largest angle of $\triangle ABC$. Finally, the network grows as $C'$ finally reaches $C$, with $CC'$ always bisecting $\angle AC'B$.*

**Proof.** Without loss of generality, assume that $\angle BAC \geq \angle ABC \geq \angle ACB$. By Lemma 3 and Theorem 6, the initial optimal network when $L = L_{\texttt{ST}} + \varepsilon$ has an equilateral triangle $A'B'C'$ growing out of the Steiner point $O$, with $AA'$, $BB'$, and $CC'$ bisecting $\angle B'A'C'$, $\angle A'B'C'$, and $\angle A'C'B'$, respectively. By Lemma 8, before $\triangle A'B'C'$ reaches $A$ as an equilateral triangle ($AA'$ is shorter than than $BB'$ and $CC'$ when $\angle BAC$ is the largest angle of $\triangle ABC$), it cannot happen that the optimal network jumps to a configuration where one anchor disappears. To see that this is the case, suppose the network jumps to a configuration where $A'$ merges with $A$. This would force $\triangle A'B'C'$ to have $\angle B'A'C' < \pi/3 < \angle A'B'C' = \angle A'C'B'$, which is not possible. The situation gets worse if $B'$ merges with $B$ or $C'$ merges with $C$. Using a similar argument, we can show that it is also not possible for the optimal network to jump from three anchors to having a single anchor without the equilateral $\triangle A'B'C'$ reaching its maximum girth. Using the same approach, we can also show that it is not possible to "jump" from a two-anchor configuration to a single anchor configuration without the anchor $B'$ reaching $B$, as the isosceles triangle expands. □

## 5.2 No Steiner Point

When an angle of $\triangle ABC$, say $\angle BAC$, is larger than $2\pi/3$, $A$ acts as a "Steiner" point. In this case, it becomes impossible for the optimal network $N(L)$ to have three internal anchors.

**Lemma 10 (Anchor Multiplicity)** *For three terminals without a Steiner point, the optimal network $N(L)$ for any $L$ cannot have three anchors.*

**Proof.** If there are three anchors, Theorem 6 must hold. However, this is impossible if one of the angles formed by the terminals is equal to or larger than $2\pi/3$. Referring to Fig. 4, suppose that $\angle BAC \geq 2\pi/3$. However, also by Theorem 6, $\angle BOC = 2\pi/3$, which is not possible. □

Following similar reasoning used for establishing the case where the Steiner point is in the interior of $\triangle ABC$, the evolution of the optimal network for the current setting goes through the following phases (assuming terminals $A$, $B$, and $C$, and $\angle BAC \geq 2\pi/3$):

1. The budget $L$ is sufficient to cover the shortest edge of $\triangle ABC$ but less than $L_{\texttt{ST}}$. In this case, $N(L)$ contains one edge of $\triangle ABC$
2. The budget $L$ equal to $L_{\texttt{ST}}$. In this case, $N(L)$ is the Steiner tree comprised of $AB$ and $AC$.
3. For $L = L_{\texttt{ST}} + \varepsilon$ for small positive $\varepsilon$, a small isosceles triangle grows out from $A$, producing a configuration as shown in Fig. 7(a). The network satisfies the bisector requirement given by Lemma 1. As $L$ increases, the isosceles triangle expands with the bisector structure in place, until one of the vertex of the triangle hits a terminal ($B$).
4. As one of the two anchors merge with a terminal, the other anchor will continue to march toward the last terminal ($C$) as $L$ increases, eventually merge with that terminal. A snapshot of this process is given in Fig. 7(b).

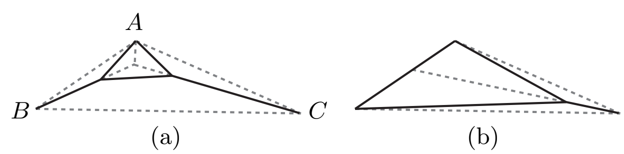


Figure 7: A spectrum of optimal Euclidean BSN network structures (solid lines) for three terminals in a typical setup where $\angle BAC \geq 2\pi/3$, as the allowed budget increases.

## 6 Conclusion and Discussions

In this work, we propose the *budgeted Steiner network* (BSN) problem to study shortest path structures among multiple terminals under a path length budget. We establish the precise evolution of the BSN structure for three arbitrarily located terminals where paths between each pair of terminals have equal importance. It is clear that the characterization yields efficient algorithms for computing optimal BSN structures for any given 3-terminal setup and length budget.

The current work just begins to scratch the surface of the study of BSN; it is natural to study the case where the weights are not equal as well as the case of more terminals. It is also interesting to explore how BSN structures are affected by obstacles. Finally, as an alternative to analytical approaches, it is interesting to explore establishing BSN structures using numerical methods.

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## 7 Appendix

**Proof.** [Proof of Lemma 4] Assume that for a given budget $L = L_{\mathtt{ST}} + \varepsilon$, the optimal network $N(L_{\mathtt{ST}} + \varepsilon)$ has corresponding optimal objective $J(L_{\mathtt{ST}} + \varepsilon)$ as given in Eq. 2. We show that $CC'$ is a bisector of $\angle A'C'B'$ by analyzing the local changes to $L$ and $J(L_{\mathtt{ST}} + \varepsilon)$ if we perturb $C'$.

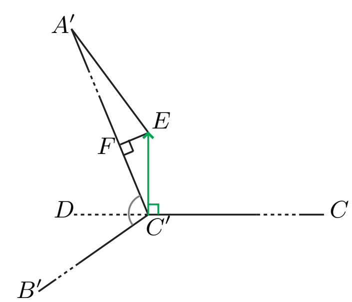


Figure 8: Perturbing $C'$ in an assumed optimal configuration for the three-terminal Euclidean BSN problem. The figure zooms in around $C'$ without showing $A$ and $B$. The drawing intentionally avoids assuming that $\triangle A'B'C'$ is an equilateral triangle.

Referring to Fig. 8, let $D$ be a point on the extension of $\overrightarrow{CC'}$. A point $E$ is introduced that shifts $C'$ up vertically (i.e., $C'E \perp C'C$) by the amount $|C'E|$, as a small perturbation to $C'$. Now draw a line $EF$ such that $EF \perp A'C'$ with $F \in A'C'$. Because $|C'E|$ is small, $|A'F| \approx |A'E|$ (this is a second order approximation). As $C'$ is moved to $E$, the length change of $A'C'$ is given by $|A'E| - |A'C'|$, which is approximately $|A'F| - |A'C'| = -|FC'| = -|C'E| \cos\angle A'C'E = -|C'E| \sin\angle A'C'D$.

Following a similar analysis procedure, the length change of $B'C'$, $|B'E| - |B'C'|$, is approximately $|C'E| \sin\angle B'C'D$. Because $C'E \perp C'C$ and $|C'E|$ is small, $|CC'| \approx |CE|$ (also a second order approximation). Relating the length changes due to moving $C'$ up to the change of the budget $L$, the net change to $L$ is $|C'E|(\sin\angle B'C'D - \sin\angle A'C'D)$ (i.e., $B'C'$ becomes longer and $A'C'$ becomes shorter with $CC'$ unchanged, as a second order approximation). The change to the objective $J(L_{\mathtt{ST}} + \varepsilon)$ is the same since $CC'$ is unaffected by $C'E$.

Because the changes to $L$ and $J(L_{\mathtt{ST}} + \varepsilon)$ are exactly the same, if $\angle A'C'D \neq \angle B'C'D$, then either $\overrightarrow{C'E}$ or a perturbation in the direction of $\overrightarrow{EC'}$ will cause both

$|A'C'|+|B'C'|+|C'C|$ and $|A'C'|+|B'C'|+2|C'C|$ to decrease, which means that $L$ and $J(L_{\mathtt{ST}}+\varepsilon)$ can be simultaneously reduced. This contradicts the assumption that $L$ is the smallest budget for which the current objective $J(L_{\mathtt{ST}}+\varepsilon)$ is possible. Since this cannot happen, it must be the case that $\angle A'C'D = \angle B'C'D$ in an optimal network configuration. That is, $CC'$ is a bisector of $\angle A'C'B'$. By symmetry, $BB'$ is a bisector of $\angle A'B'C'$ and $AA'$ is a bisector of $\angle B'A'C'$. □

**Proof.** [Proof of Theorem 6] Again assuming an optimal solution, extend line segments $AA'$, $BB'$, and $CC'$ so that they intersect (see Fig. 9).

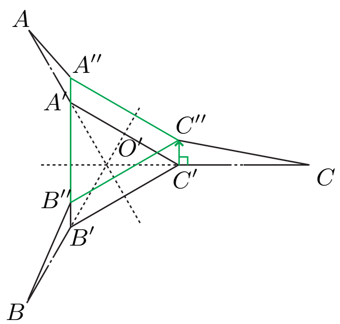


Figure 9: Applying a perturbation to $\triangle A'B'C'$ that lifts it vertically along $CC'$, which keeps the length of $CC'$ unchanged in a first order approximation.

Because they are bisectors of $\triangle A'B'C'$, by Lemma 4, they must meet at the same point $O'$. For this setting, we again apply a perturbation argument used in proving Lemma 4, this time lift the entire $\triangle A'B'C'$ in a direction perpendicular to $CC'$. Let the perturbed triangle be $\triangle A''B''C''$. Using the same argument, this time applied to the length changes of $AA'$ and $BB'$, we can reach the conclusion that the line $CC'$ must be a bisector of $\angle AO'B$. In other words, shifting $AA'$ and $BB'$ synchronously will not reduce the objective function only if $CC'$ bisects $\angle AO'B$.

Similarly, $AA'$ must be a bisector of $BO'C$ and $BB'$ must be a bisector of $AO'C$. Using that $CC'$ bisects $AO'B$ and $A'C'B'$, it can be derived that $\angle O'A'C' = \angle O'B'C'$, which in turn shows that $\angle B'A'C' = \angle A'B'C'$. By symmetry, it can then be concluded that $\triangle A'B'C'$ is an equilateral triangle. This further shows that $\angle A'O'B' = \angle A'O'C' = \angle B'O'C' = 2\pi/3$, implying that $O'$, the center of $\triangle A'B'C'$, is the Steiner point $O$ of the terminals. □

**Proof.** [Proof of Proposition 1] By Lemma 5, $BB'$ bisects $\angle AB'C'$ and $CC'$ bisects $\angle AC'B'$. Let the extensions of $BB'$ and $CC'$ meet at $O'$ (see Fig. 5). Then $AO'$ bisects $\angle B'AC'$. Using the perturbation argument from the proof of Theorem 6, applied to perturb the lengths of $BB'$ and $CC'$, we can show that $AO'$ is also a bisector of $\angle B'O'C'$ (we do this by "rotating" $\triangle AB'C'$ with center $A$ slightly). This means that $\angle B'O'A = \angle C'O'A$, which in turn implies that $\angle AB'O' = \angle AC'O'$ and further implies $\angle AB'C' = \angle AC'B'$. Therefore, $\triangle AB'C'$ is an isosceles triangle and $AB' = AC'$. □