On the Biplanar and k-Planar Crossing Numbers

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Abstract

The biplanar crossing number of a graph G is the minimum number of crossings over all possible drawings of the edges of G in two disjoint planes. We present new bounds on the biplanar crossing number of complete graphs and complete bipartite graphs. In particular, we prove that the biplanar crossing number of complete bipartite graphs can be approximated to within a factor better than 3, improving over the best previously known approximation factor of 4.03. For complete graphs, we prove an approximation factor of 3.17, improving the best previously known factor of 4.34. We provide similar improved bounds for the k-planar crossing number of complete graphs and complete bipartite graphs, for any positive integer k.

1 Introduction

An embedding (or drawing) of a graph G in the Euclidean plane is a mapping of the vertices of G to distinct points in the plane and a mapping of edges to smooth curves between their corresponding vertices. A planar embedding of a graph is a drawing of the graph in the plane such that edges intersect only at their endpoints. A graph admitting such a drawing is called planar. A *biplanar embedding* of a graph $G = (V, E)$ is a decomposition of the graph into two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, together with planar embeddings of G_1 and G_2. In this case, we call G biplanar. Biplanar embeddings are central to the computation of thickness of graphs [13], with applications to VLSI design [14]. It is well-known that planarity can be recognized in linear time, while biplanarity testing is NP-complete [12].

Let $cr(G)$ be the minimum number of edge crossings over all drawings of G in the plane, and let $cr_k(G)$ be the minimum of $cr(G_1) + \cdots + cr(G_k)$ over all possible decompositions of G into k subgraphs $G_1, \ldots, G_k$. We call $cr(G)$ the *crossing number* of G, and $cr_k(G)$ the *k-planar crossing number* of G. Throughout this paper, we only consider *simple drawings* for each subgraph G_i, in which no two edges intersect more than once, and no three edges intersect at a point (such drawings are sometimes called nice drawings). Moreover, we denote by n the number of vertices, and by m the number of edges of a graph.

Determining the crossing number of complete graphs and complete bipartite graphs has been the subject of extensive research over the past decades. In 1955, Zarankiewicz [20] conjectured that the crossing number $cr(K_{p,q})$ of the complete bipartite graph $K_{p,q}$ is equal to

$$Z(p,q) := \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor.$$

He also established a drawing with that many crossings. In 1960, Guy [8] conjectured that the crossing number $cr(K_n)$ of the complete graph K_n is equal to

$$Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Both conjectures have remained open after more than six decades. For the biplanar case, even formulating such conjectures seems to be hard. As noted in [4], techniques like embedding method and the bisection width method which are useful for bounding ordinary crossing numbers do not seem applicable to the biplanar case.

In 1971, Owens [14] described a biplanar embedding of K_n with almost $\frac{7}{24}Z(n)$ crossings. The construction was later improved by Durocher *et al.* [7], but the upper bound remained asymptotically the same. In 2006, Czabarka *et al.* [4] presented a biplanar embedding for $K_{p,q}$ with about $\frac{2}{9}Z(p,q)$ crossings. They also proved that $cr_2(K_n) \geq n^4/952$ and $cr_2(K_{p,q}) \geq p(p-1)q(q-1)/290$. Shahrokhi *et al.* [17] generalized these lower bounds to the k-planar case. Pach *et al.* [15] proved that for every graph G and any positive integer k, $cr_k(G) \leq \left(\frac{2}{k^2} - \frac{1}{k^3}\right) cr(G)$. This includes as a special case the inequality $cr_2(G) \leq \frac{3}{8}cr(G)$, originally proved by Czabarka *et al.* [5].

Our results. In this paper, we present several new bounds for approximating the biplanar and k-planar crossing number of complete graphs and complete bipartite graphs. Given a positive integer k and a real constant $\alpha \geq 1$, we say that $cr_k(K_n)$ is *approximated* to within a factor of α, if there is an upper bound $f(n)$ and a lower bound $g(n)$ on the value of $cr_k(K_n)$ such that $\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq \alpha$. Here, α is called an *asymptotic approximation factor* for $cr_k(K_n)$. Similarly, we

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say that $cr_k(K_{p,q})$ is approximated to within a factor of α, if there is an upper bound $f(p,q)$ and a lower bound $g(p,q)$ on the value of $cr_k(K_{p,q})$ such that $\lim_{p,q\to\infty} \frac{f(p,q)}{g(p,q)}$ exists and is no more than α. The results presented in this paper are summarized below.

- We prove that for all $p,q \geq 30$, $cr_2(K_{p,q}) \geq p(p-1)q(q-1)/213$. This significantly improves the best current lower bound of $cr_2(K_{p,q}) \geq p(p-1)q(q-1)/290$, due to Czabarka *et al.* [4]. Combined with the upper bound of $cr_2(K_{p,q}) \leq \frac{2}{9}Z(p,q)+o(p^2q^2)$[1] [4], our result implies an asymptotic approximation factor of 2.96 for $cr_2(K_{p,q})$, improving over the best previously known asymptotic factor of 4.03.

- For complete graphs, we show that $cr_2(K_n) \geq \frac{n^4}{694}$, improving the best current lower bound of $cr_2(K_n) \geq \frac{n^4}{952}$ [4]. Combined with the upper bound of $cr_2(K_n) \leq \frac{7}{24}Z(n) + o(n^4)$ due to Owens [14], we achieve an asymptotic approximation factor of 3.17 for $cr_2(K_n)$, improving the best previously known approximation factor of 4.34.

- We extend our lower bounds for the biplanar crossing number to the k-planar case, for any positive integer k. In particular, we show that for sufficiently large n, $cr_k(K_n) \geq n^4/(232k^2)$, improving the best current lower bound of $cr_k(K_n) \geq n^4/(432k^2)$, due to Shahrokhi *et al.* [17]. Considering the upper bound of $cr_k(K_n) \leq \frac{2}{k^2}Z(n)$ due to Pach *et al.* [15], we obtain an asymptotic approximation factor of 7.25 for $cr_k(K_n)$, improving the best current approximation factor of 13.5 available for $cr_k(K_n)$.

- Finally, we prove that for any positive integer k, $cr_k(K_{p,q}) \geq p(p-1)q(q-1)/(73.2k^2)$, improving the current lower bound of $cr_k(K_{p,q}) \geq p(p-1)q(q-1)/(108k^2)$ due to Shahrokhi *et al.* [17]. Combined with the upper bound of $cr_k(K_n) \leq \frac{2}{k^2}Z(p,q)$ [15], we obtain an asymptotic approximation factor of 9.15 for $cr_k(K_{p,q})$, improving the best current factor of 13.5.

A summary of the asymptotic approximation factors for the biplanar and k-planar crossing number of K_n and $K_{p,q}$ is presented in Table 1.

2 Two Combinatorial Lemmas

We first present two combinatorial lemmas which are the main ingredients of our proofs. Our first lemma shows how we can derive a lower bound on the k-planar crossing number of a graph G based on a lower bound on the (ordinary) crossing number of that graph, if G belongs to a family of graphs closed under edge removal, such as simple graphs and bipartite graphs.

Table 1: Summary of asymptotic approximation factors for the biplanar and k-planar crossing numbers.

Crossing Number	Asymptotic Approx. Factor	Ref.
$cr_2(K_{p,q})$	4.03 **2.96**	[4] [This work]
$cr_2(K_n)$	4.34 **3.17**	[4, 14] [This work]
$cr_k(K_{p,q})$	13.5 **9.15**	[15, 17] [This work]
$cr_k(K_n)$	13.5 **7.25**	[15, 17] [This work]

Lemma 1 *Let $\mathcal{G}$ be a hereditary class of graphs which is closed under removing edges. Let $f(x) = \alpha x$, for some positive constant α, and let $g(x)$ be an arbitrary function of x. If for every graph G in $\mathcal{G}$, $cr(G) \geq f(m) - g(n)$, then $cr_k(G) \geq f(m) - k \cdot g(n)$ for all $G \in \mathcal{G}$ and for all positive integers k.*

Proof. Fix a graph $G \in \mathcal{G}$. Let $G = \bigcup_{i=1}^{k} G_k$ be a decomposition of G into k subgraphs $G_i = (V, E_i)$ such that $\sum_{i=1}^{k} cr(G_i)$ is minimum. By the hereditary property of $\mathcal{G}$, each G_i is a member of $\mathcal{G}$, and hence $cr(G_i) \geq f(m_i) - g(n)$, where $m_i = |E_i|$. Therefore, $cr_k(G) = \sum_{i=1}^{k} cr(G_i) \geq \sum_{i=1}^{k}(f(m_i) - g(n)) = \alpha \sum_{i=1}^{k} m_i - \sum_{i=1}^{k} g(n) = f(m) - k \cdot g(n)$. □

Another combinatorial tool typically used for deriving lower bounds on the crossing number of graphs is the counting method (see, e.g., [9, 16]). We use the following generalization of the counting method in this paper.

Lemma 2 (Counting method) *Let G be a simple graph that contains α copies of a subgraph H. If in every k-planar drawing of G, each crossing of the edges belongs to at most β copies of H, then*

$$cr_k(G) \geq \left\lceil \frac{\alpha}{\beta} cr_k(H) \right\rceil .$$

Proof. Let D be a k-planar drawing of G, realizing $cr_k(G)$. For each of the α copies of H, D contains a k-planar drawing with at least $cr_k(H)$ crossings. Since each crossing is counted at most β times by our assumption, the lemma statement follows. Note that a ceiling is put in the right-hand side of the inequality, because $cr_k(G)$ is always an integer. □

[1]By definition, $f(x,y) = o(g(x,y))$ if $\lim_{x,y\to\infty} \frac{f(x,y)}{g(x,y)} = 0$.

3 Lower Bounds for Complete Bipartite Graphs

In this section, we provide new lower bounds on the biplanar crossing number of complete bipartite graphs. In particular, we improve the following bound due to Czabarka *et al.* [4] which states that for all $p, q \geq 10$,

$$cr_2(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{290}.$$

From Euler's formula, we have $cr(G) \geq m - 3(n-2)$ for simple graphs, and $cr(G) \geq m - 2(n-2)$ for bipartite graphs. Using Lemma 1, we immediately get a lower bound of $cr_2(G) \geq m - 6(n-2)$ for simple graphs, and a lower bound of $cr_2(G) \geq m - 4(n-2)$ for bipartite graphs.

To establish stronger lower bounds, we need to incorporate more powerful ingredients. A graph is called k-planar, if it can be drawn in the plane in such a way that each edge has at most k crossings. It is known that every 1-planar drawing of a 1-planar graph has at most $n-2$ crossings [6]. (Note the difference between k-planar graphs, and k-planar crossing numbers.) Removing one edge per crossing yields a planar graph. Therefore, every 1-planar bipartite graph has at most $3n-6$ edges. Karpov [10] proved that for every 1-planar bipartite graph with at least 4 vertices, the inequality $m \leq 3n-8$ holds. In a recent work, Angelini *et al.* [2] proved that for every 2-planar bipartite graph we have $m \leq 3.5n - 7$. We use these results to obtain the following stronger lower bound.

Lemma 3 *For every bipartite graph G with $n \geq 4$,*

$$cr_k(G) \geq 3m - (8.5n - 19)k.$$

Proof. Let G be a bipartite graph with n vertices and m edges. Fix a drawing of G with a minimum number of crossings. If $m > 3.5n - 7$, then by [2], there must be an edge in the drawing with at least three crossings. We repeatedly remove such an edge until we reach a drawing with $\lfloor 3.5n - 7 \rfloor$ edges. Now, by Karpov's result, there must be an edge in the drawing with at least two crossings. We repeatedly remove such an edge until we reach a drawing with $3n-8$ edges. Let G' be the bipartite graph corresponding to the remaining drawing. We know by Euler's formula that $cr(G') \geq (3n-8) - 2(n-2)$. Therefore,

$$\begin{aligned} cr(G) &\geq 3(m - \lfloor 3.5n - 7 \rfloor) + 2(\lfloor 3.5n - 7 \rfloor - (3n-8)) \\ &\quad + (3n-8) - 2(n-2) \\ &\geq 3m - \lfloor 3.5n - 7 \rfloor - (3n-8) - 2(n-2) \\ &\geq 3m - 8.5n + 19. \end{aligned}$$

Applying Lemma 1 yields $cr_k(G) \geq 3m - (8.5n - 19)k$. □

For complete bipartite graphs, Lemma 3 implies that $cr_2(K_{p,q}) \geq 3pq - 17(p+q) + 38$, for all $p, q \geq 2$. We use Lemma 3 along with a counting argument to obtain the following improved bound on $cr_2(K_{p,q})$.

Theorem 4 *For all $p, q \geq 30$,*

$$cr_2(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{213}.$$

Proof. Using the counting method (Lemma 2) for $K_{p,p}$ and $K_{p+1,p}$ we have

$$cr_2(K_{p+1,p}) \geq \left\lceil \frac{p+1}{p-1} cr_2(K_{p,p}) \right\rceil.$$

This is because $K_{p+1,p}$ contains $p+1$ copies of $K_{p,p}$, and each crossing realized by two edges, belongs to at most $\binom{p-1}{p-2} = p-1$ of these copies. Using a similar argument for $K_{p+1,p}$ and $K_{p+1,p+1}$, we get

$$cr_2(K_{p+1,p+1}) \geq \left\lceil \frac{p+1}{p-1} \left\lceil \frac{p+1}{p-1} cr_2(K_{p,p}) \right\rceil \right\rceil. \quad (1)$$

By Lemma 3, $cr_2(K_{15,15}) \geq 203$. Plugging into (1), yields $cr_2(K_{16,16}) \geq 266$, Now, we use the recurrence relation (1) iteratively from $p = 16$ to 30 to get

$$cr_2(K_{30,30}) \geq 3554. \quad (2)$$

We can now apply the counting method on $K_{30,30}$ and $K_{p,q}$ to obtain

$$\begin{aligned} cr_2(K_{p,q}) &\geq \frac{\binom{p}{30}\binom{q}{30}}{\binom{p-2}{28}\binom{q-2}{28}} \; cr_2(K_{30,30}) \\ &= \frac{p(p-1)q(q-1)}{30 \times 29 \times 30 \times 29} \; cr_2(K_{30,30}). \end{aligned}$$

Plugging (2) in the above inequality yields the theorem statement. □

Remark. The exact value of the denominator obtained in the above proof is around 212.97. One may continue applying the recurrence relation (1) to obtain better bounds for $K_{p,p}$, when $p > 30$. This leads to a slightly improved constant in the denominator, but it does not seem to reduce the constant below 212. Indeed, the denominator seems to converge to a value around 212.4, for large values of p.

4 Biplanar Crossing Number of Complete Graphs

We now consider the biplanar crossing number of complete graphs. Czabarka *et al.* [4] used a probabilistic method to prove that for large values of n,

$$cr_2(K_n) \geq \frac{n^4}{952}.$$

We improve this lower bound using the counting method.

Theorem 5 *For all* $n \geq 24$,

$$cr_2(K_n) \geq \frac{n(n-1)(n-2)(n-3)}{698}.$$

Proof. We know from [1] that for every G with $n \geq 3$, $cr(G) \geq 5m - \frac{139}{6}(n-2)$. Applying Lemma 1, we get

$$cr_2(G) \geq 5m - \frac{139}{3}(n-2).$$

This in particular implies $cr_2(K_{25}) \geq 435$. Now, we use the counting method (Lemma 2) on K_{25} and K_n to get

$$cr_2(K_n) \geq \frac{\binom{n}{25} cr_2(K_{25})}{\binom{n-4}{21}} \geq \frac{n(n-1)(n-2)(n-3)}{\frac{25\times24\times23\times22}{435}},$$

which implies the theorem statement. □

We can slightly improve this result, using an iterative counting method similar to what we used in the previous section.

Theorem 6 *For large values of* n,

$$cr_2(K_n) \geq \frac{n^4}{694}.$$

Proof. Using the counting method (Lemma 2) for K_n and K_{n+1} we have

$$cr_2(K_{n+1}) \geq \left\lceil \frac{(n+1)cr_2(K_n)}{n-3} \right\rceil. \quad (3)$$

Starting from $cr_2(K_{25}) \geq 435$, we use the recurrence relation (3) iteratively from $n = 25$ to 50 to obtain $cr_2(K_{50}) \geq 7965$. Now, we use the counting method on K_{50} and K_n to get

$$\begin{aligned} cr_2(K_n) &\geq \frac{\binom{n}{50} cr_2(K_{50})}{\binom{n-4}{46}} \\ &\geq \frac{n(n-1)(n-2)(n-3)}{\frac{50\times49\times48\times47}{7965}} \\ &\geq \frac{n(n-1)(n-2)(n-3)}{693.94}, \end{aligned}$$

which implies $cr_2(K_n) \geq \frac{n^4}{694}$ for sufficiently large n. □

5 k-Planar Crossing Number of K_n and $K_{p,q}$

In this section, we provide improved lower bounds on the k-planar crossing number of complete bipartite and complete graphs. Shahrokhi *et al.* [17] proved that for any positive integer k, and sufficiently large integers p, q, and n:

$$cr_k(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{108k^2},$$

and

$$cr_k(K_n) \geq \frac{n(n-1)(n-2)(n-3)}{432k^2}.$$

We improve these results using the ideas developed in Sections 3 and 4.

Theorem 7 *For all* $p, q \geq 8k+2$,

$$cr_k(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{73.2k^2}.$$

Proof. We apply the counting method (Lemma 2) on $K_{8k+2,8k+2}$ and $K_{p,q}$. By Lemma 3, for every bipartite graph G, $cr_k(G) \geq 3m - (8.5n - 19)k$. This yields

$$cr_k(K_{8k+2,8k+2}) \geq 56k^2 + 43k + 12.$$

Hence,

$$\begin{aligned} cr_k(K_{p,q}) &\geq \frac{\binom{p}{8k+2}\binom{q}{8k+2} cr_k(K_{8k+2,8k+2})}{\binom{p-2}{8k}\binom{q-2}{8k}} \\ &= \frac{p(p-1)q(q-1)cr_k(K_{8k+2,8k+2})}{(8k+2)(8k+1)(8k+2)(8k+1)} \\ &\geq \frac{p(p-1)q(q-1)}{\frac{(8k+2)^2(8k+1)^2}{56k^2+43k+12}} \\ &\geq \frac{p(p-1)q(q-1)}{\frac{512}{7}k^2}, \end{aligned}$$

which completes the proof. □

Theorem 8 *For all* $n \geq 14k-3$,

$$cr_k(K_n) \geq \frac{n(n-1)(n-2)(n-3)}{232k^2}.$$

Proof. We use the counting method (Lemma 2) for K_{14k-3} and K_n. Recall that for every G with $n \geq 3$, $cr(G) \geq 5m - \frac{139}{6}(n-2)$ [1]. Therefore, $cr_k(G) \geq 5m - \frac{139}{6}(n-2)k$ by Lemma 1. Thus,

$$cr_k(K_{14k-3}) \geq \frac{497}{3}k^2 - \frac{775}{6}k + 30.$$

Therefore,

$$\begin{aligned} cr_k(K_n) &\geq \frac{\binom{n}{14k-3} cr_k(K_{14k-3})}{\binom{n-4}{14k-7}} \\ &= \frac{n(n-1)(n-2)(n-3)cr_k(K_{14k-3})}{(14k-3)(14k-4)(14k-5)(14k-6)}, \end{aligned}$$

which implies the theorem. □

6 Conclusion

In this paper, we presented several improved bounds on the biplanar and k-planar crossing number of complete graphs and complete bipartite graphs. An obvious open problem is whether the asymptotic approximation factors presented in this paper can be further improved. Obtaining similar bounds on the k-planar crossing number of other graph classes is an intriguing open problem.

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