Globally linked pairs in braced maximal outerplanar graphs

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Abstract

We say that a graph G is a braced MOP graph if it contains a maximal outerplanar graph as a spanning subgraph. We show that a pair $\{u, v\}$ of vertices of a braced MOP graph is globally linked in $\mathbb{R}^2$ in every generic realization of G if and only if uv is an edge of G or G contains three pairwise openly disjoint u-v paths. It follows that a braced MOP graph is globally rigid if and only if it is 3-vertex connected or isomorphic to K_3.

The former result verifies the conjectured characterization of global linkedness in the plane in a special case. The latter result leads to a linear time algorithm for testing global rigidity of braced MOP graphs.

Our proof is based on new structural results about maximal outerplanar graphs.

1 Introduction

Given a set $V = \{v_1, v_2, \ldots, v_n\}$ of n labeled points, a map $p : V \to \mathbb{R}^d$ defines a point configuration $P = \{p_1, p_2, \ldots, p_n\}$ of V in $\mathbb{R}^d$, where $p_i = p(v_i)$, for $1 \leq i \leq n$. For a subset E of the pairs of points in V, a basic geometric problem is whether every d-dimensional point configuration Q of V for which the pairwise distances in P and in Q are the same for all pairs in E, is congruent with P. In the local version of this question we focus on a pair v_i, v_j and ask whether the distance between the points corresponding to this pair is the same as in P in every point configuration Q that agrees with P on E. It is known that these decision problems are NP-hard, even in $\mathbb{R}^1$ [17].

If we restrict ourselves to generic point configurations, then global uniqueness, up to congruence, depends only on E (for fixed d). A combinatorial characterization of those pairs (V, E) that give rise to globally unique generic point configurations is known for $d = 1, 2$. The higher dimensional cases are still open. We also have a similar combinatorial question concerning the local version, which is open for all $d \geq 2$.

In this paper we consider the latter problem in $\mathbb{R}^2$ and characterize local uniqueness for a new family of graphs. Our results verify a conjectured characterization (Conjecture 1 below) in a special case. These problems are best described and studied by using the notions and tools of rigidity theory. In the rest of this section we introduce these notions as well as previous results of this area.

1.1 Globally linked pairs in frameworks and graphs

A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We also say that (G, p) is a *realization* of G in $\mathbb{R}^d$. Intuitively, we can think of a framework (G, p) as a collection of bars and joints where each vertex v of G corresponds to a joint located at $p(v)$ and each edge to a rigid (that is, fixed length) bar joining its endpoints. Two frameworks (G, p) and (G, q) are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. A pair of frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$. This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of $\mathbb{R}^d$. A framework (G, p) is said to be *generic* if the set of its $d|V(G)|$ vertex coordinates is algebraically independent over $\mathbb{Q}$.

A d-dimensional framework (G, p) is called *globally rigid* if every equivalent d-dimensional framework (G, q) is congruent to (G, p). The framework (G, p) is *rigid* if there exists some $\varepsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(v) - q(v)|| < \varepsilon$ for all $v \in V$, then (G, q) is congruent to (G, p). This is equivalent to requiring that every continuous motion of the vertices of (G, p) in $\mathbb{R}^d$ that preserves the edge lengths takes the framework to a congruent realization of G. It is known that for generic frameworks the rigidity (resp. global rigidity) in a given dimension depends only on G: either every generic realization of G in $\mathbb{R}^d$ is (globally) rigid, or none of them are [1, 4]. Thus, we say that a graph G is *rigid* (resp. *globally rigid*) in $\mathbb{R}^d$ if every (or equivalently, if some) d-dimensional generic realization of G is globally rigid in $\mathbb{R}^d$.

For $d = 1, 2$, combinatorial characterizations and corresponding deterministic polynomial time algorithms

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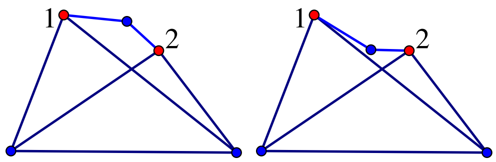

Figure 1: Two equivalent generic realizations of a graph G in $\mathbb{R}^2$. The graph is not globally rigid, but the pair $\{1,2\}$ is globally linked in G.

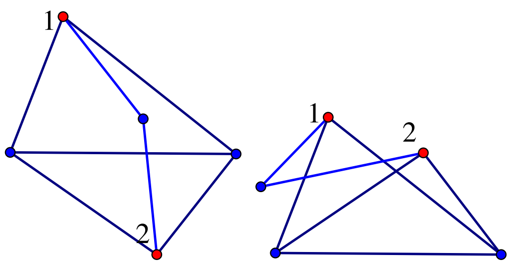

Figure 2: Two equivalent generic realizations of the same graph G in which $\{1,2\}$ is not globally linked. The lengths of the edges incident with 1 and 2 permit a partial reflection in the rest of the graph.

are known for (testing) rigidity in $\mathbb{R}^d$ as well as global rigidity in $\mathbb{R}^d$ [7, 14, 16]. The existence of such a characterization (or algorithm) for $d \geq 3$ is a major open question in both cases. For more details on (globally) rigid graphs and frameworks see e.g. [12, 18].

A pair of vertices $\{u,v\}$ in a framework (G,p) is *globally linked in* (G,p) if for every equivalent framework (G,q) we have $||p(u)-p(v)|| = ||q(u)-q(v)||$. The pair $\{u,v\}$ is *globally linked in* G in $\mathbb{R}^d$ if it is globally linked in all generic d-dimensional frameworks (G,p). It is immediate from the definitions that G is globally rigid in $\mathbb{R}^d$ if and only if all pairs of vertices of G are globally linked in G in $\mathbb{R}^d$. Global linkedness in $\mathbb{R}^d$ is not a generic property (for $d \geq 2$): a vertex pair may be globally linked in some generic d-dimensional realization of G without being globally linked in all generic realizations. See Figures 1, 2.

The case $d = 1$ is exceptional and well-understood: a pair is globally linked in G in $\mathbb{R}^1$ if and only if there is a cycle in G that contains both vertices. In higher dimensions, no combinatorial (or efficiently testable) characterization is known for globally linked pairs in graphs.

1.2 The rigidity matroid

The rigidity matroid of a graph G is a matroid defined on the edge set of G which reflects the rigidity properties of all generic realizations of G. For a general introduction to matroid theory we refer the reader to [15]. For a detailed treatment of the 2-dimensional rigidity matroid, see [11].

Let (G,p) be a realization of a graph $G=(V,E)$ in $\mathbb{R}^d$. The *rigidity matrix* of the framework (G,p) is the matrix $R(G,p)$ of size $|E| \times d|V|$, where, for each edge $uv \in E$, in the row corresponding to uv, the entries in the d columns corresponding to vertices u and v contain the d coordinates of $(p(u)-p(v))$ and $(p(v)-p(u))$, respectively, and the remaining entries are zeros. The rigidity matrix of (G,p) defines the *rigidity matroid* of (G,p) on the ground set E by linear independence of the rows. It is known that any pair of generic frameworks (G,p) and (G,q) have the same rigidity matroid. We call this the d-dimensional *rigidity matroid* $\mathcal{R}_d(G) = (E, r_d)$ of the graph G.

We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. A graph $G=(V,E)$ is $\mathcal{R}_d$*-independent* if $r_d(G) = |E|$ and it is an $\mathcal{R}_d$*-circuit* if it is not $\mathcal{R}_d$-independent but every proper subgraph G' of G is $\mathcal{R}_d$-independent. We note that in the literature such graphs are sometimes called M-independent in $\mathbb{R}^d$ and M-circuits in $\mathbb{R}^d$, respectively. An edge e of G is an $\mathcal{R}_d$*-bridge in* G if $r_d(G-e) = r_d(G) - 1$ holds. Equivalently, e is an $\mathcal{R}_d$-bridge in G if it is not contained in any subgraph of G that is an $\mathcal{R}_d$-circuit.

The following characterization of rigid graphs is due to Gluck.

Theorem 1 *[3] Let $G=(V,E)$ be a graph with $|V| \geq d+1$. Then G is rigid in $\mathbb{R}^d$ if and only if $r_d(G) = d|V| - \binom{d+1}{2}$.*

A graph is *minimally rigid* in $\mathbb{R}^d$ if it is rigid in $\mathbb{R}^d$ but deleting any edge results in a flexible graph. By Theorem 1, minimally rigid graphs in $\mathbb{R}^d$ on at least $d+1$ vertices have exactly $d|V| - \binom{d+1}{2}$ edges.

Let $\mathcal{M}$ be a matroid on ground set E. We can define a relation on the pairs of elements of E by saying that $e, f \in E$ are equivalent if $e = f$ or there is a circuit C of $\mathcal{M}$ with $\{e,f\} \subseteq C$. This defines an equivalence relation. The equivalence classes are the *connected components* of $\mathcal{M}$. The matroid is *connected* if there is only one equivalence class, and *separable* otherwise. A graph $G=(V,E)$ is $\mathcal{R}_d$*-connected* if $\mathcal{R}_d(G)$ is connected. The subgraphs induced by the edges of the connected components of $\mathcal{R}_d(G)$ are the $\mathcal{R}_d$*-components* of G. Thus an edge $e \in E$ is an $\mathcal{R}_d$-bridge if $\{e\}$ is a trivial $\mathcal{R}_d$-component of G.

In the next section we shall see that $\mathcal{R}_2$-connected graphs play an important role in the characterization of globally rigid graphs in $\mathbb{R}^2$ (see Theorem 2 below) as well as in the conjectured characterization of globally linked pairs in $\mathbb{R}^2$.

2 Previous work

In the rest of the paper, we focus on the $d = 2$ case. Thus, we shall occasionally write that a graph is (globally) rigid to mean that it is (globally) rigid in $\mathbb{R}^2$, and we may similarly omit the dimension when referring to global linkedness of vertex pairs in graphs. The following characterization of globally rigid graphs in $\mathbb{R}^2$ (the equivalence of (i) and (ii) below) is from [7]. As it was noted in [9], (ii) is in fact equivalent to (iii).

Theorem 2 *[7] Let G be a graph on at least four vertices. The following assertions are equivalent.*

(i) G is globally rigid in $\mathbb{R}^2$,
(ii) G is 3-connected and $\mathcal{R}_2$-connected,
(iii) G is 3-connected and contains no $\mathcal{R}_2$-bridges.

The complete characterization of globally linked pairs of vertices in a graph in $\mathbb{R}^2$ is not known. The truth of the following conjecture would imply a complete answer and an efficient algorithm for testing global linkedness.

Let $H = (V, E)$ be a graph and $x, y \in V$. We use $\kappa_H(x, y)$ to denote the maximum number of pairwise internally disjoint xy-paths in H. Note that if $xy \notin E$ then, by Menger's theorem, $\kappa_H(x, y)$ is equal to the size of a smallest set $S \subseteq V(H) - \{x, y\}$ for which there is no xy-path in $H - S$. It is easy to see that if $\kappa_G(x, y) \leq 2$ and $xy \notin E(G)$ then $\{x, y\}$ is not globally linked in G in $\mathbb{R}^2$ [9].

Conjecture 1 *[9, Conjecture 5.9] The pair $\{x, y\}$ is globally linked in a graph $G = (V, E)$ in $\mathbb{R}^2$ if and only if either $xy \in E$ or there is an $\mathcal{R}_2$-component H of G with $\{x, y\} \subseteq V(H)$ and $\kappa_H(x, y) \geq 3$.*

The following partial results are known. The first characterizes globally linked pairs in $\mathcal{R}_2$-connected graphs and implies the "if" direction of Conjecture 1.

Theorem 3 *[9, Theorem 5.7] Suppose that G is $\mathcal{R}_2$-connected and let u, v be a pair of vertices in G. Then $\{u, v\}$ is globally linked in G in $\mathbb{R}^2$ if and only if $\kappa_G(x, y) \geq 3$.*

In the case of minimally rigid graphs (in which every $\mathcal{R}_2$-component is trivial, i.e. isomorphic to K_2) it has been shown that there are no non-adjacent globally linked pairs.

Theorem 4 *[10] Let $G = (V, E)$ be a minimally rigid graph in $\mathbb{R}^2$ and $u, v \in V$. Then $\{u, v\}$ is globally linked in G in $\mathbb{R}^2$ if and only if $uv \in E$.*

3 Maximal outerplanar graphs

A (simple) graph $G = (V, E)$ on $n \geq 3$ vertices is said to be *outerplanar* if it has a planar embedding in which all vertices of G lie on the boundary of the outer face. It is well-known that if G is a 2-connected outerplanar graph, then G has a unique Hamiltonian cycle C, which bounds the outer face in any in such planar embedding of G. We shall consider 2-connected outerplanar graphs and call this cycle C the *boundary cycle* of G. The edges in $E - E(C)$ are called the *diagonals* of G.

A *maximal outerplanar graph*, or *MOP graph*, for short, is an outerplanar graph G for which $G + uv$ is not outerplanar for all nonadjacent vertex pairs $u, v \in V$. Equivalently, G is the graph of the triangulation of a polygon. Note that every MOP graph on at least three vertices is 2-vertex connected: indeed, if v is a cut-vertex in the outerplanar graph G, then G necessarily has some pair u_1, u_2 of non-adjacent neighbours for which $G + u_1u_2$ is outerplanar.

It is well-known that every 2-connected outerplanar graph has at least two vertices of degree two. For a degree two vertex v of a MOP graph on at least four vertices $G - v$ is also a MOP graph. Since K_3 is minimally rigid and adding a new vertex of degree two preserves minimal rigidity (see e.g. [11]), we have the following observation.

Proposition 5 *Every MOP graph is minimally rigid in $\mathbb{R}^2$.*

We shall mainly be concerned with the following larger family of graphs. A graph G' is a *braced MOP graph* if it contains a spanning MOP subgraph. If we fix such a subgraph $G = (V, E)$, so that $G' = (V, E \cup B)$, then we say that the edges in B are the *braces* of the graph. If $|B| = 1$ (i.e. if G' has $2|V| - 2$ edges), then we say that G' is a *uni-braced MOP graph.*

In order to identify the globally linked pairs in braced MOP graphs we first obtain new structural results concerning the connectivity properties of MOP graphs.

3.1 Separating pairs and 3-connected subgraphs in (braced) MOP graphs

A pair $\{u, v\}$ of vertices of G is a *separating pair* if $G - \{u, v\}$ is disconnected. The structure of the separating pairs in a MOP graph is rather special.

Lemma 6 *Let $G = (V, E)$ be a MOP graph and let $\{u, v\} \subset V$. Then $\{u, v\}$ is a separating pair if and only if uv is a diagonal of G.*

Proof. The "if" part follows from planarity (and holds for all 2-connected outerplanar graphs). To prove the "only if" direction suppose that $\{u, v\}$ is a separating pair in G. Let C denote the boundary cycle of G. Since C is a 2-connected spanning subgraph of G, the vertices u and v cannot be consecutive on C. Thus if $uv \in E$ then uv is a diagonal and we are done. Let us suppose, for a contradiction, that $uv \notin E$. It is clear that $G - \{u, v\}$ has exactly two connected components A_1, A_2 whose vertex sets coincide with the vertex sets of the two paths P_1, P_2 obtained from C by deleting u and v. The fact that G is outerplanar and there are no

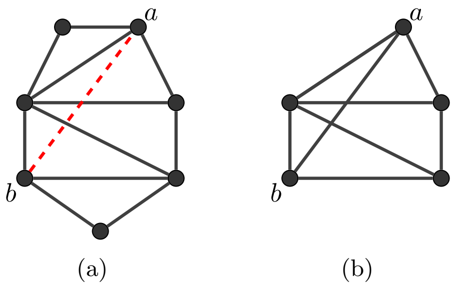

Figure 3: (a) A uni-braced MOP graph G' with a spanning MOP subgraph G (indicated by the solid edges) and a bracing edge ab. (b) the graph $H_G[a,b] + ab$.

edges in G between P_1 and P_2 imply that $G + uv$ is also outerplanar, contradicting the maximality of G. □

Consider a MOP graph $G = (V, E)$ and two non-adjacent vertices $a, b \in V$. We say that a diagonal uv is (a, b)*-separating* if a and b are in different connected components of $G - \{u, v\}$.

Proposition 7 *Let $G = (V, E)$ be a MOP graph and let $a, b \in V$ be two non-adjacent vertices. Then*
(i) for every (a, b)-separating diagonal v_iv_j and two internally disjoint a-b-paths P_1, P_2 we have (after relabeling the paths, if necessary) $v_i \in V(P_1)$, $v_j \in V(P_2)$,
(ii) for every diagonal v_kv_l which is not (a, b)-separating and for the component D of $G - \{v_k, v_l\}$ disjoint from $\{a, b\}$ we have that $G - V(D)$ is a MOP graph. In particular, $\kappa_{G-V(D)}(a, b) \geq 2$.

Proof. Lemma 6 implies (i). Part (ii) follows from the observation that for every diagonal v_iv_j and component K of $G - \{v_i, v_j\}$, $G - V(K)$ is a MOP graph. □

By using Proposition 7(ii) to "peel off" vertex sets from G, preserving 2-connectivity, we can deduce the following.

Lemma 8 *Let $G = (V, E)$ be a MOP graph and let $a, b \in V$ be two non-adjacent vertices. Then there is a unique smallest 2-connected subgraph H of G that contains a, b. This subgraph H is a MOP graph, induced by the vertex set consisting of the end-vertices of the (a, b)-separating diagonals, plus a, b.*

Proof. Let X be the set of the end-vertices of the (a, b)-separating diagonals in G. By Proposition 7(i) the vertex set of every 2-connected subgraph of G that contains a, b must also contain X. Let us fix an outerplanar embedding of G and consider a minimal (with respect to inclusion) subgraph of G which induces a MOP subgraph in this embedding and contains a, b. It follows from Proposition 7(ii) that every diagonal v_kv_l in H is (a, b)-separating, for otherwise we can delete the vertices of the component D of $H - \{v_k, v_l\}$ which is disjoint from $\{a, b\}$ to obtain a smaller MOP subgraph. The key observation is that such a deletion does not create new (a, b)-separating diagonals. Hence the vertex set of H must be equal to $X \cup \{a, b\}$. This completes the proof. □

The unique subgraph H in Lemma 8 belonging to a, b in G is denoted by $H_G[a, b]$. Note that $H_G[a, b]$ has exactly two vertices of degree two: a and b.

We close this subsection with a lemma which may be of independent interest. Since we shall not use it later, we omit the proof. (Necessity follows from Lemma 6.)

Lemma 9 *Let $G = (V, E)$ be a 2-connected graph. Then G is a MOP graph if and only if for all non-adjacent vertex pairs $\{a, b\} \subset V$ there is a separating pair $\{u, v\}$ with $uv \in E$ for which $G - \{u, v\}$ has exactly two connected components, each of which contains exactly one of a and b.*

3.2 Globally rigid braced MOP graphs

Lemma 10 *Let $G = (V, E)$ be a MOP graph and let $a, b \in V$ be two non-adjacent vertices. Then $H_G[a, b] + ab$ is a 3-connected $\mathcal{R}_2$-circuit.*

Proof. Since $H_G[a, b]$ is a MOP graph, every separating pair consists of the end-vertices of a diagonal by Lemma 6. Moreover, every diagonal in $H_G[a, b]$ is (a, b)-separating by Lemma 8. Hence $H_G[a, b] + ab$ is 3-connected.

By Proposition 5, $H_G[a, b]$ is minimally rigid. Thus, by basic matroid theory, $H_G[a, b] + ab$ contains a unique $\mathcal{R}_2$-circuit W with $ab \in E(W)$. It is well-known that deleting any edge of an $\mathcal{R}^2$-circuit results in a minimally rigid graph (see e.g. [7, Lemma 2.15]), so in particular $W - ab$ is minimally rigid. Since minimally rigid graphs on at least three vertices are 2-connected, the minimality of $H_G[a, b]$ implies that $W - ab = H_G[a, b]$ holds. Hence $H_G[a, b] + ab$ is an $\mathcal{R}_2$-circuit, as claimed. □

The above discussion leads to the following result.

Lemma 11 *Let G' be a uni-braced MOP graph obtained from the MOP graph G by adding the brace ab. Then $H_G[a, b] + ab$ is a maximal globally rigid subgraph of G'.*

Proof. The global rigidity of $H_G[a, b] + ab$ follows from Lemma 10 and Theorem 2. Maximality is due to the fact that every vertex v of G' which is disjoint from $H_G[a, b] + ab$ is separated from $H_G[a, b] + ab$ by a diagonal. □

Our first main result is as follows.

Theorem 12 *Let G' be a braced MOP graph. Then G' is globally rigid if and only if G' is 3-connected (or $G = K_3$).*

Proof. The only (braced) MOP graph and the only globally rigid graph on three vertices is K_3, so we may assume that $|V| \geq 4$. Necessity follows from Theorem 2. To prove sufficiency, we shall verify that every edge of G' belongs to an $\mathcal{R}_2$-circuit. Let us fix a spanning MOP subgraph $G = (V, E)$ and let B denote the set of braces of G'. For a brace $ab \in B$, Lemma 10 implies that $H_G[a, b] + ab$ is an $\mathcal{R}_2$-circuit that contains ab.

Next, consider a diagonal uv of G. Since G' is 3-connected, there is a brace $a'b' \in B$ that connects the two connected components of $G - \{u, v\}$. Then $\{u, v\}$ is (a', b')-separating, so $H[a', b'] + a'b'$ is an $\mathcal{R}_2$-circuit that contains uv by Lemmas 8 and 10.

Finally, consider an edge $e = v_iv_{i+1}$ of the boundary cycle C of G. The 3-connectivity of G' implies that the degree of v_i and v_{i+1} in G' is at least three. Hence there is a diagonal v_iv_k incident with v_i. Let us suppose that v_k is as close to v_{i+1} as possible on the path $C - \{v_i\}$. By using 3-connectivity again, we can see that there is a brace cd that connects the two connected components of $G - \{v_i, v_k\}$. We may suppose that d and v_{i+1} are in the same components. Now the choice of v_k and the MOP structure implies that the $\mathcal{R}_2$-circuit $H_G[c, d] + cd$ contains the edge e.

Since G' is 3-connected and contains no $\mathcal{R}_2$-bridges, it is globally rigid by Theorem 2. □

3.3 Globally linked pairs in braced MOP graphs

We next consider the globally linked pairs. The next general lemma is easy to verify. Let H_1, H_2 be two disjoint graphs on at least three vertices with two designated edges $e_i = u_iv_i$, $e_i \in E(H_i)$, $i = 1, 2$. We say that the graph G obtained from H_1, H_2 by identifying u_1 with u_2 and v_1 with v_2 is the 2-*merge* of H_1 and H_2 along the edges e_i, $i = 1, 2$.

Lemma 13 *Let $G = (V, E)$ be the 2-merge of H_1 and H_2 and let $x, y \in V$. Then $\{x, y\}$ is globally linked in G if and only if $\{x, y\} \subseteq V(H_i)$ and $\{x, y\}$ is globally linked in H_i for some $i \in \{1, 2\}$.*

Our second main result is as follows.

Theorem 14 *Let G' be a braced MOP graph, obtained from the MOP graph $G = (V, E)$ by adding a set B of braces, and let $x, y \in V$. Then $\{x, y\}$ is globally linked in G' if and only if either $xy \in E(G')$ or $\kappa_{G'}(x, y) \geq 3$.*

Proof. We may assume that G' has at least four vertices. Suppose that $xy \notin E(G')$. Then $\kappa_{G'}(x, y) \geq 3$ is a necessary condition for the global linkedness of $\{x, y\}$ in G' (for otherwise applying a suitable partial reflection to a generic realization of G' shows that x and y are not globally linked). To prove sufficiency suppose that we have $\kappa_{G'}(x, y) \geq 3$. Every separating pair of G' is a separating pair of G. Thus every separating pair in G' consists of the end-vertices of a diagonal of G by Lemma 8. Hence we may assume, by Lemma 13, that G' is a 3-connected braced MOP graph. Then the theorem follows from Theorem 12. □

From the proof it also follows that if $\{x, y\}$ is globally linked in the braced MOP graph G', then there is an $\mathcal{R}_2$-component H of G with $x, y \in V(H)$. Indeed, the proof shows that there is even a globally rigid subgraph H' with $x, y \in V(H')$, and globally rigid graphs in $\mathbb{R}^2$ are $\mathcal{R}_2$-connected, c.f. Theorem 2. Thus, Theorem 14 also implies Conjecture 1 in the case of braced MOP graphs.

We note that there exist $\mathcal{R}_2$-connected graphs which are not braced MOP graphs (for example the complete bipartite graph $K_{3,4}$), as well as braced MOP graphs which are not $\mathcal{R}_2$-connected (for example, K_4 plus a degree-two vertex). Thus neither of Theorem 3 and Theorem 14 is implied by the other.

We can also deduce the following results on braced MOP graphs. For a 2-dimensional realization (G, p), let $r(G, p)$ denote the number of pairwise non-congruent realizations of G that are equivalent to (G, p). A *globally rigid cluster* of G is a maximal vertex set of G in which each vertex pair is globally linked in G.

Theorem 15 *Let G be a braced MOP graph. Then*

(i) global linkedness is a generic property of G, that is, for each vertex pair $\{x, y\}$ either the pair is globally linked in every generic realization of G, or not globally linked in every generic realization of G,

(ii) for every generic realization (G, p) of G in $\mathbb{R}^2$ we have $r(G, p) = 2^{c(G)}$, where $c(G)$ is the number of diagonals uv of G for which $\{u, v\}$ is a separating pair in G',

(iii) a vertex set X in G' is a globally rigid cluster if and only if $G[X]$ is a maximal globally rigid subgraph of G' if and only if $G[X]$ is a maximal 3-connected subgraph of G' (or a K_3).

We close this section with an inductive construction of globally rigid uni-braced MOP graphs. A result in [7] shows that every globally rigid graph can be obtained from K_4 by a sequence of so-called 1-extensions and edge additions. Our construction uses the vertex splitting operation, which is one of the frequently used tools in the theory of rigid (planar) graphs, see e.g. [2]. Let uv be an edge of the graph G and let F_1, F_2 be a partition of the edges incident with v in $G - uv$. The *vertex splitting* operation at vertex v on the edge uv consists of deleting the vertex v from G and then adding two new vertices v_1, v_2 and new edges uv_1, uv_2, v_1v_2, as well as the edges xv_i for every edge xv with $xv \in F_i$, $i = 1, 2$. See Figure 4. The vertex splitting operation is said to be *non-trivial* if F_1, F_2 are both non-empty.

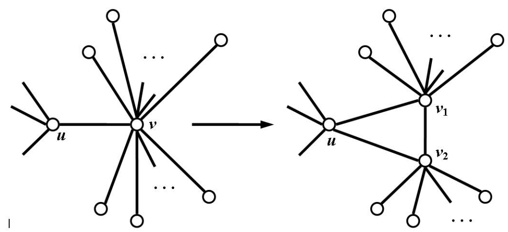

Figure 4: The vertex splitting operation at vertex v on edge uv.

Let v be a vertex of a MOP graph G, uv a diagonal incident with v, a let C_1, C_2 be the connected components of $G-\{u,v\}$. Then an *outerplanar vertex splitting* operation at v on uv is a vertex splitting operation in which F_i consists of the edges that connect v to C_i, $i = 1, 2$. It is not difficult to see that a graph obtained by an outerplanar vertex splitting from a MOP graph is again a MOP graph. In a uni-braced MOP graph G' with spanning MOP subgraph G and brace ab, we can define an outerplanar vertex splitting operation at v on edge uv in a similar manner, provided ab is disjoint from v.

With these definitions in place, we can show the following:

Theorem 16 *A graph is a 3-connected uni-braced MOP graph if and only if it can be obtained from K_4 by a sequence of non-trivial outerplanar vertex splitting operations.*

Proof. It is easy to see that non-trivial vertex splitting preserves 3-connectivity. Thus all graphs obtained from K_4 by non-trivial outerplanar vertex splitting operations are 3-connected uni-braced MOP graphs. To see the other direction, we show that if G' is such a graph on at least five vertices then we can perform the inverse operation (i.e. the contraction of some edge vv' of the boundary cycle which is disjoint from the brace and for which v and v' have a common neighbour) so that we obtain a smaller 3-connected uni-braced MOP graph. Let ab be the brace in G' and let $G = (V, E)$ be the underlying MOP graph. Every diagonal of G is (a, b)-separating, since otherwise the end-vertices of the diagonal would form a separating pair in G' by Lemma 6. Since G has at least five vertices, the boundary cycle of G must contain an edge $e = vv'$ that is disjoint from $\{a, b\}$. By the MOP property v and v' have a common neighbour. Thus e is contractible in the above sense and hence the resulting graph G'' obtained from G' by contracting e is a uni-braced MOP graph with brace ab. Moreover, G'' is 3-connected, since every diagonal of G is (a, b)-separating. The theorem follows by induction on the number of vertices. □

4 Questions and conjectures

Although we have extended the family of those graphs for which global linkedness is well characterized and computationally tractable in $\mathbb{R}^2$, Conjecture 1 remains open. On the other hand, our results motivate the following new questions:

(1) For which family of (rigid) graphs is global linkedness in $\mathbb{R}^2$ a generic property?

Examples of families that satisfy this property are $\mathcal{R}_2$-connected graphs (Theorem 3), graphs with a spanning MOP graph (Theorem 15), so-called special graphs (i.e. minimally rigid graphs without proper non-complete rigid subgraphs) [9] and 2-trees (graphs that can be obtained from a sequence of 2-merge operations on triangles).

(2) For which family of (rigid) graphs is global linkedness in $\mathbb{R}^2$ characterized by the $\kappa \geq 3$ condition (for non-adjacent vertices)?

The next question may also be interesting.

(3) For which family of (rigid) graphs does global linkedness in $\mathbb{R}^2$ hold only for adjacent vertex pairs?

Here we offer the following conjecture, which would extend Theorem 4, and which would follow from Conjecture 1.

Conjecture 2 *A graph $G = (V, E)$ has no globally linked pairs $\{u, v\}$ in $\mathbb{R}^2$ with $uv \notin E$ if and only if every $\mathcal{R}_2$-connected component of G is either a complete graph or can be obtained from a sequence of 2-merge operations on complete graphs of size at least four.*

Necessity follows from Theorem 3.

5 Concluding remarks

We have characterized the globally linked pairs in braced MOP graphs in $\mathbb{R}^2$, extending the family of graphs for which the statement of Conjecture 1 holds. As a by-product we have obtained a simpler characterization for the two-dimensional global rigidity of braced MOP graphs which makes it possible to test their global rigidity in linear time, by using fast algorithms that can check if a graph is 3-connected, see e.g. [6]. We note that the problem of testing, in polynomial time, whether a given graph is a braced MOP graph seems to be open.

The theory of globally rigid graphs and globally linked pairs has found numerous applications, including wireless sensor network localization, see [8]. Our result can also be used in this context. For example, the characterization of globally linked pairs (Theorem 14) enables one to identify the uniquely localizable vertices in $\mathbb{R}^2$ with respect to any designated set of so-called anchor vertices in a sensor network, provided the graph of the network contains a spanning MOP graph. It also implies an affirmative answer to another conjecture of [9] concerning unique localizability in $\mathbb{R}^2$, when the graph is a braced MOP.

A recent research direction is to find efficient algorithms that can add a smallest set of new edges to a graph so that it becomes globally rigid, see [13]. By Theorem 12 this augmentation problem can be reduced to the well-studied 3-connectivity augmentation problem of graphs, provided the input is a (braced) MOP graph.

Finally, we remark that we have been able to substantially generalize our characterization of globally linked pairs in MOP graphs in $\mathbb{R}^2$ in several ways. In particular, the analogues of Theorems 12 and 14 turn out to also hold, in $d \geq 1$ dimensions, for so-called "braced d-trees", and, more generally, "braced d-connected chordal graphs." These new results will be part of a forthcoming manuscript.

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