A bound for Delaunay flip algorithms on flat tori

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Abstract

We are interested in triangulations of flat tori. A Delaunay flip algorithm performs Delaunay flips on the edges of an input triangulation T until it reaches a Delaunay triangulation. We prove that no sequence of Delaunay flips is longer than $C_\Gamma \cdot n^2 \cdot \Lambda(T)$ where $\Lambda(T)$ is the maximum length of an edge of T, n is the number of vertices of T, and $C_\Gamma > 0$ depends only on the flat torus. The bound improves on the upper bound previously known [4] in three ways: the dependency in the "quality" of the input triangulation is linear instead of quadratic, the bound is tight, and the "quality parameter" is simpler.

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1 Introduction

Delaunay triangulations are mostly known in the Euclidean plane setting. In this context a triangulation T can be defined as a maximal planar subdivision of a finite set of points P [3, Chapter 9]. If the two bounded faces of T incident to an inner edge e form a strictly convex quadrilateral then the edge e can be replaced, in T, by the other diagonal of the quadrilateral. Such operation is called a flip. The flip graph of P is the graph whose vertices are the triangulations on P and such that two triangulations are linked by an edge if there is a flip transforming one into the other. This graph is connected and its diameter is quadratic in the cardinal of P [5]. A triangulation is Delaunay if the circumdisk of every bounded face contains no point of the triangulation in its interior. A Delaunay flip algorithm takes as input a triangulation and performs Delaunay-flips until it reaches a Delaunay triangulation. Such an algorithm terminates [3, Observation 9.3].

Generalizing Delaunay triangulations [2] [1] and Delaunay flip algorithms [4] to other geometric spaces than the Euclidean plane is a natural question that has been studied (and implemented [7] [6]). In that setting Delaunay flip algorithms present the advantage of handling triangulations containing loops and multi-edges. A flat torus $\mathbb{T}_\Gamma$ is the quotient space of the Euclidean plane under the action of a group Γ generated by two independent translations (Section 2.1). In this paper we are interested in the complexity (number of flips) of Delaunay flip algorithms on flat tori. We prove Theorem 1.

Theorem 1 *Every sequence of Delaunay flips starting from a triangulation T of a flat torus $\mathbb{T}_\Gamma$ has length at most*

$$C_\Gamma \cdot n^2 \cdot \Lambda(T)$$

where $\Lambda(T)$ is the maximum length of an edge of T, n is the number of vertices of T, and $C_\Gamma > 0$ depends only on $\mathbb{T}_\Gamma$. This bound is asymptotically tight.

An upper bound was already proved [4, Theorem 16], together with the connectivity of the flip graph, as a particular (easy) case of a more general result on geometric triangulations of hyperbolic surfaces:

$$C_h \cdot n^2 \cdot \Delta(T)^2$$

where C_h depends only on $\mathbb{T}_\Gamma$ and $\Delta(T)$ is a parameter measuring in some sense how "stretched" T is. The definition of $\Delta(T)$ is not used in this paper but we give it (in the special case of triangulations of flat tori) for the interested reader: the real $\Delta(T)$ is the smallest diameter that can have a domain of $\mathbb{R}^2$ that is the union over every face t of the triangulation T of a lift (Section 2.1) of the face t.

To obtain their bound the authors showed that the edges flipped in a sequence of Delaunay flips cannot be longer than $2\Delta(T)$ [4, Lemma 10]. The upper bound follows from the observation that the number of segments no longer than $L > 0$ between two given points of $\mathbb{T}_\Gamma$ is at most quadratic in L.

Our first (small) improvement is to replace the parameter $\Delta(T)$ by the maximum length $\Lambda(T)$ of an edge in T. The inequality $\Lambda(T) \leq \Delta(T)$ is easily observed to be true. Moreover the definition of $\Delta(T)$ is more intricate than the definition of $\Lambda(T)$ and it is not obvious how to compute $\Delta(T)$.

Our second (main) improvement is to replace the quadratic dependency by a linear dependency in $\Lambda(T)$, obtaining a bound that is asymptotically tight.

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2 Background

In this paper $\mathbb{R}^d$, $d \geq 1$, denotes the usual d-dimensional Euclidean space with the L_2 norm. We call *segment* of $\mathbb{R}^d$ the convex hull $[\widetilde{u}, \widetilde{v}]$ of any two distinct points $\widetilde{u}, \widetilde{v} \in \mathbb{R}^d$. We call *interior* of $[\widetilde{u}, \widetilde{v}]$ the set $[\widetilde{u}, \widetilde{v}] \setminus \{\widetilde{u}, \widetilde{v}\}$. The interior of a segment of $\mathbb{R}^d$ is not empty.

2.1 Flat tori

A *flat torus* $\mathbb{T}_\Gamma$ is the quotient of $\mathbb{R}^2$ under the action of a group Γ generated by two independent translations. For the needs of this section we introduce the projection $\rho : \mathbb{R}^2 \to \mathbb{T}_\Gamma$ mapping every point of $\mathbb{R}^2$ to its Γ-orbit.

We call *segment* of $\mathbb{T}_\Gamma$ any projection $s = \rho(\widetilde{s})$ of a segment $\widetilde{s}$ of $\mathbb{R}^2$ such that the restriction of ρ to the interior of $\widetilde{s}$ is injective. If $\widetilde{u}$ and $\widetilde{v}$ are the endpoints of $\widetilde{s}$ then $\rho(\widetilde{u})$ and $\rho(\widetilde{v})$ are the (possibly equal) *endpoints* of s. We call *interior* of s the image by ρ of the interior of $\widetilde{s}$.

A *lift* of a point $p \in \mathbb{T}_\Gamma$ is any point $\widetilde{p}$ in the Γ-orbit $\rho^{-1}(p)$. A lift of a segment s of $\mathbb{T}_\Gamma$ is any segment $\widetilde{s}$ of $\mathbb{R}^2$ whose interior is, through ρ, in one-to-one correspondence with the interior of s.

The *length* $l(s)$ of a segment s of $\mathbb{T}_\Gamma$ is the length of a lift of s in $\mathbb{R}^2$. It is independent of the lift.

2.2 Delaunay triangulations and flip algorithms

A topological triangulation of a flat torus $\mathbb{T}_\Gamma$ is any embedding of a finite undirected graph onto $\mathbb{T}_\Gamma$ such that each resulting face is homeomorphic to an open disk and is bounded by exactly three distinct edge-embeddings. Observe that this graph may have loops or multiple edges. A geometric triangulation of $\mathbb{T}_\Gamma$ is a topological triangulation in which each edge is embedded as a segment of $\mathbb{T}_\Gamma$. In this paper every triangulation is geometric so we just use the term *triangulation.*

The lift of a triangulation T of $\mathbb{T}_\Gamma$ is the infinite triangulation of $\mathbb{R}^2$ whose vertices and edges are the lifts of the vertices and edges of T.

A *Delaunay triangulation* of $\mathbb{T}_\Gamma$ is a triangulation T of $\mathbb{T}_\Gamma$ whose lift $\widetilde{T}$ is a Delaunay triangulation of $\mathbb{R}^2$ (Figure 1). In other words for each face $\widetilde{t}$ of $\widetilde{T}$ the disk circumscribing $\widetilde{t}$ contains no vertex of $\widetilde{T}$ in its interior. We refer to the literature for an introduction to Delaunay triangulations of $\mathbb{R}^2$ [3, Chapter 9].

Consider a triangulation T of $\mathbb{T}_\Gamma$, an edge e of T and a lift $\widetilde{e}$ of e. The segment $\widetilde{e}$ of $\mathbb{R}^2$ is an edge of the lift $\widetilde{T}$ of T and $\widetilde{e}$ is incident with two faces $\widetilde{t_1}$ and $\widetilde{t_2}$ of $\widetilde{T}$. Let $\widetilde{D_1}$ and $\widetilde{D_2}$ be the open disks of $\mathbb{R}^2$ circumscribing $\widetilde{t_1}$ and $\widetilde{t_2}$ respectively. Let also $\widetilde{v}_1$ be the vertex of $\widetilde{t_1}$ that is not a vertex of $\widetilde{t_2}$, and $\widetilde{v}_2$ be the vertex of $\widetilde{t_2}$ that is not a vertex of $\widetilde{t_1}$. The condition $\widetilde{v}_1 \in \widetilde{D_2}$ is equivalent to $\widetilde{v}_2 \in \widetilde{D_1}$. If it is satisfied we say that the edge e is *Delaunay-flippable* in the triangulation T and this definition is independent of the choice of the lift $\widetilde{e}$. In such a case the union of the closures of $\widetilde{t_1}$ and $\widetilde{t_2}$ is a convex quadrilateral and replacing, in the triangulation T, the edge e by the segment $\rho([\widetilde{v}_1, \widetilde{v}_2])$ of $\mathbb{T}_\Gamma$ yields another triangulation T' of $\mathbb{T}_\Gamma$. We say that the triangulation T' results from the *Delaunay flip* of the edge e in the triangulation T.

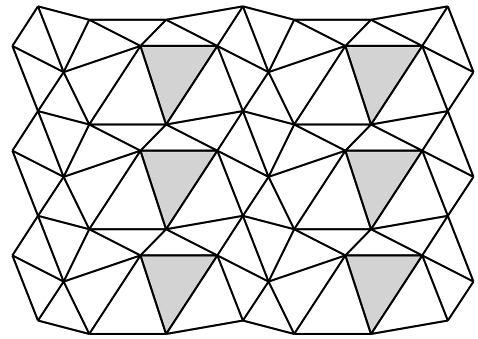

Figure 1: A portion of the lift of a Delaunay triangulation of a flat torus. (Gray) Six lifts of a single face.

We call *sequence of Delaunay flips* any sequence $T_0, \ldots, T_m$ of triangulations of $\mathbb{T}_\Gamma$, for some $m \geq 0$, such that for every $k \in \{1, \ldots, m\}$ the triangulation T_k results from the Delaunay flip of an edge in the triangulation T_{k-1}. We say that m is the *length* of the sequence.

Every *Delaunay flip algorithm* takes as input a triangulation of $\mathbb{T}_\Gamma$ and flips Delaunay-flippable edges until there is none left to flip. Such an algorithm terminates and outputs a Delaunay triangulation [4].

2.3 Stereographic projection and Delaunay flips

In $\mathbb{R}^3$ let $\mathbb{S}_2$ denote the 2-dimensional sphere of radius 1 centered at $(0,0,0)$. The point $P = (0,0,-1)$ belongs to $\mathbb{S}_2$. We identify $\mathbb{R}^2$ with the plane of $\mathbb{R}^3$ containing the points whose third coordinate is 1. Given $\widetilde{p} \in \mathbb{R}^2$ we denote by $I_{\widetilde{p}}$ the unique line of $\mathbb{R}^3$ containing the points $\widetilde{p}$ and P (Figure 2).

The *stereographic projection* π is a bijection from $\mathbb{R}^2$ to $\mathbb{S}_2 \setminus P$. It maps every point $\widetilde{p} \in \mathbb{R}^2$ to the unique intersection of the line $I_{\widetilde{p}}$ with $\mathbb{S}_2 \setminus P$.

A triangle in $\mathbb{R}^3$ is the convex hull of three points that do not belong to a common line. We call *triangular surface* any connected union of triangles satisfying the following properties. Firstly if the intersection of any two distinct triangles of the union is not empty then it is either a vertex or an edge of both of the two triangles. Secondly every edge belongs to at most two triangles. Finally the triangles incident to a common vertex v can be either circularly or linearly ordered so that two such triangles share an edge e that is incident to v if and only if the two triangles are adjacent in the (circular or linear) ordering.

Every infinite triangulation $\mathcal{T}$ of $\mathbb{R}^2$ is mapped uniquely to a triangular surface S as follows. The vertices of S are the images of the vertices of $\mathcal{T}$ under π and the triangles of S are in one-to-one correspondence with the faces of $\mathcal{T}$: the three vertices $\widetilde{v_1}, \widetilde{v_2}$ and $\widetilde{v_3}$ of a face of $\mathcal{T}$ are mapped to the three vertices $\pi(\widetilde{v_1}), \pi(\widetilde{v_2})$, and $\pi(\widetilde{v_3})$ of a triangle of S. We say that such a triangular surface (issued of an infinite triangulation of $\mathbb{R}^2$) is *standard*.

We emphasize that every standard triangular surface shares no other point with the sphere $\mathbb{S}_2$ than its vertices. In fact if a point belongs to, but is not a vertex of, a standard triangular surface then it is at distance less than one from the point $(0, 0, 0)$.

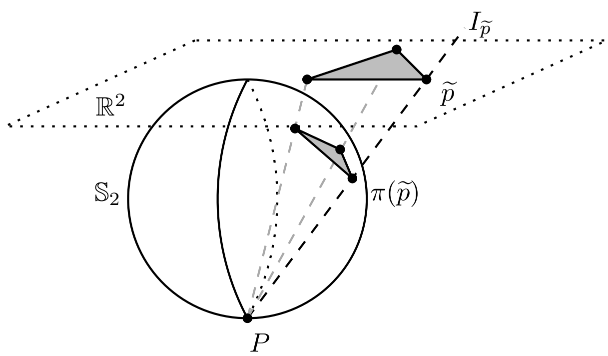

Figure 2: Mapping a lift of a triangulation of flat torus to a standard triangular surface.

Every standard triangular surface S induces bijection $\pi_S : \mathbb{R}^2 \to S$ sending every $\widetilde{p} \in \mathbb{R}^2$ to the unique intersection with S of the line $I_{\widetilde{p}}$. Given two standard triangular surfaces S and S' (possibly with $S = S'$) we say that S *is above* S' if for every $\widetilde{p} \in \mathbb{R}^2$ the point $\pi_{S'}(\widetilde{p})$ lies on the closed segment $[P, \pi_S(\widetilde{p})]$ of $\mathbb{R}^3$, on the line $I_{\widetilde{p}}$. The aboveness relation is a partial order on the set of standard triangular surfaces. Lemma 2 is folklore and follows from the fact that every circle on $\mathbb{R}^2$ is mapped under the stereographic projection to a circle on $\mathbb{S}_2 \setminus P$, the latter being the intersection with $\mathbb{S}_2 \setminus P$ of a plane of $\mathbb{R}^3$.

Lemma 2 *Assume that a triangulation T of a flat torus $\mathbb{T}_\Gamma$ results from the Delaunay flip of an edge e' in a triangulation T' of $\mathbb{T}_\Gamma$ and let e be the edge of T resulting from the flip. Let S and S' be the standard triangular surfaces associated to the lifts of T and T', respectively. Then S is above S'. Let also $p \in \mathbb{T}_\Gamma$ be the intersection point of the interiors of e and e' and $\widetilde{p} \in \mathbb{R}^2$ be any lift of p. Then $\pi_S(\widetilde{p}) \neq \pi_{S'}(\widetilde{p})$.*

3 Lower bound

On a flat torus $\mathbb{T}_\Gamma$ the length of a sequence of Delaunay flips ending at a Delaunay triangulation cannot be bounded from above by a function depending only on the number of vertices of the starting triangulation. This fact follows from two observations. The first observation is that it is easy to construct an infinite set of triangulations of $\mathbb{T}_\Gamma$ all having a single common vertex, say v, as their vertex set (Figure 3). The second observation is that there can only be a finite number of Delaunay triangulations of $\mathbb{T}_\Gamma$ having v as their unique vertex[1].

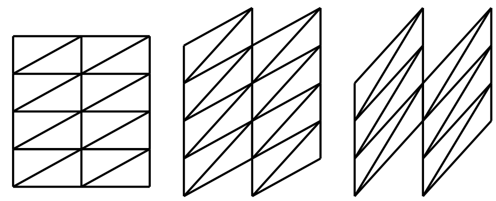

Figure 3: On a flat torus, three portions of the lifts of three triangulations with a single common vertex.

To understand this phenomenon more precisely, we consider a second parameter of the starting triangulation T: the maximum length $\Lambda(T)$ of an edge in T. We exhibit in Proposition 3 a family of starting triangulations T for which we prove a lower bound on the length of every sequence of Delaunay flips starting from T and ending at a Delaunay triangulation.

We are interested in a particular flat torus. Consider the two independent translations by the vectors $(1, 0)$ and $(0, 1)$ respectively. We are interested in the flat torus $\mathbb{T}_\square$ that is the quotient of $\mathbb{R}^2$ under the action of the group generated by those two translations. We denote by $\rho_\square$ the canonical projection from $\mathbb{R}^2$ to $\mathbb{T}_\square$. We say that $\mathbb{T}_\square$ is the *unit flat torus*.

Proposition 3 *For every $n \geq 1$ and every $\Lambda_0 > 0$ there is a triangulation T of the unit flat torus $\mathbb{T}_\square$ such that every sequence of Delaunay flips starting from T and ending at a Delaunay triangulation is longer than*

$$c \cdot n^2 \cdot \Lambda(T)$$

where $\Lambda(T) > \Lambda_0$ is the maximum length of an edge in T, n is the number of vertices of T, and $c > 0$ is a constant.

The quadratic dependence in the number of vertices is also a consequence of a more general fact about flips (not necessarily Delaunay flips) of triangulated polygons in the plane [5, Theorem 3.8]. Our construction is inspired from one previously known in that setting [5].

[1] Pick's theorem [8] infers the existence of $\Lambda_1 > 0$ depending only on $\mathbb{T}_\Gamma$ such that in $\mathbb{R}^2$ every disk of diameter Λ_1 intersects a lift of v. It follows that the edges of any Delaunay triangulation of $\mathbb{T}_\Gamma$ with vertex set $\{v\}$ are not longer than Λ_1. There can only be a finite number of such edges.

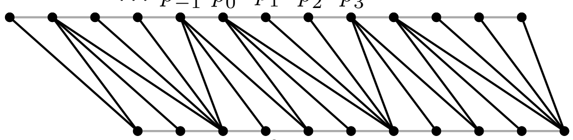

Figure 4: A portion of the lift of a triangulation belonging to $\mathcal{F}$ in the proof of Proposition 3. The fixed edges are in gray.

Proof. We fix $n \geq 1$ and $\Lambda_0 > 0$. See Figure 4.

For every $z \in \mathbb{Z}$ and every $\epsilon \in \{0, 1\}$ we define the point $\widetilde{p}^\epsilon_z = (\frac{z}{n}, \epsilon)$ in $\mathbb{R}^2$ and the point p_z of $\mathbb{T}_\square$ by $p_z = \rho_\square(\widetilde{p}^0_z)$. Observe that if $z, z' \in \mathbb{Z}$ are such that $z \equiv z'$ mod n then $p_z = p_{z'}$ and the points $\widetilde{p}^0_z, \widetilde{p}^1_z, \widetilde{p}^0_{z'}$, and $\widetilde{p}^1_{z'}$ are all lifts of p_z. For every $z, z' \in \mathbb{Z}$ we define the segment $s_{z,z'}$ of $\mathbb{T}_\square$ as $\rho_\square([\widetilde{p}^0_z, \widetilde{p}^1_{z'}])$.

We are interested in the set $\mathcal{F}$ of the triangulations of $\mathbb{T}_\square$ satisfying the following. The vertices of every triangulation $T \in \mathcal{F}$ are $p_1, \dots, p_n$ and the edges of T are partitioned as follows: T contains n edges that we call *fixed* and $2n$ edges that we call *free*. For $k \in \{1, \dots, n\}$ the k^{th} fixed edge of T is $\rho_\square([\widetilde{p}^0_{k-1}, \widetilde{p}^0_k])$. The only restriction on the free edges of T is that they must belong to $\{s_{z,z'} \ : \ z, z' \in \mathbb{Z}\}$.

Claim 1. For every $T \in \mathcal{F}$ the following holds:

(a) The fixed edges of T are not Delaunay-flippable.

(b) The Delaunay flip of a free edge e in T results in a triangulation $T' \in \mathcal{F}$.

(c) Such a Delaunay flip replaces the edge e in T by an edge e' in T' such that $l(e') \geq l(e) - 2/n$.

(d) The lengths of two free edges of T cannot differ by more than 2.

Claim 2. There is a triangulation in $\mathcal{F}$ having a free edge longer than Λ_0.

Claim 3. There is a constant $\Lambda_1 > 0$ such that the edges of every Delaunay triangulation in $\mathcal{F}$ are not longer than Λ_1.

Claims 2 and 3 are straightforward. We will prove Claim 1 in the end. We first show that those claims imply the result. By Claim 2 there is a triangulation $T_0 \in \mathcal{F}$ having a free edge longer than Λ_0. Let $\Lambda(T)$ denote the maximum length of an edge in T_0; $\Lambda(T)$ is the length of a free edge of T_0. Indeed the free edges of T_0 have length at least 1 while the fixed edges of T_0 have length $1/n$.

We assign to every triangulation $T \in \mathcal{F}$ a weight $\omega(T)$ that is the sum of the lengths of its edges. By Claim 1.d $\omega(T_0) \geq 1 + 2n(\Lambda(T) - 2)$. Indeed T_0 has n fixed edges of length $1/n$ and $2n$ free edges of length at least $\Lambda(T) - 2$.

Consider a sequence $T_0, \dots, T_m$ of Delaunay flips for some $m \geq 0$ that starts from T_0 and ends at a Delaunay triangulation T_m. By Claims 1.a and 1.b all the triangulations $T_0, \dots, T_m$ belong to $\mathcal{F}$. By Claim 1.c holds $\omega(T_m) \geq \omega(T_0) - 2m/n$. By Claim 3 there is a constant $\Lambda_1 > 0$ such that $\omega(T_m) \leq 3n\Lambda_1$. Thus

$$2m \geq n(\omega(T_0) - \omega(T_m)) \geq n + (2\Lambda(T) - 3\Lambda_1 - 4)n^2.$$

That proves the result. Now we prove Claim 1.

Proof of Claim 1. To prove (a) consider a fixed edge e of the triangulation T. There is $k \in \{1, \dots, n\}$ such that the segment $\widetilde{e}$ of $\mathbb{R}^2$ between $\widetilde{p}^0_{k-1} = (\frac{k-1}{n}, 0)$ and $\widetilde{p}^0_k = (\frac{k}{n}, 0)$ is a lift of e. Consider the two faces $\widetilde{t_1}$ and $\widetilde{t_2}$ of the lift $\widetilde{T}$ of T that are incident to $\widetilde{e}$. Let $\widetilde{v}_1$ be the vertex of $\widetilde{t_1}$ that is not a vertex of $\widetilde{t_2}$ and let $\widetilde{v}_2$ be the vertex of $\widetilde{t_2}$ that is not a vertex of $\widetilde{t_1}$. Up to renaming $\widetilde{v}_1$ and $\widetilde{v}_2$ there are $z, z' \in \mathbb{Z}$ such that $\widetilde{v}_1 = \widetilde{p}^1_z = (\frac{z}{n}, 1)$ and $\widetilde{v}_2 = (\frac{z'}{n}, -1)$. It is straightforward to check that the open disk whose boundary contains $\widetilde{p}^0_{k-1}$,$\widetilde{p}^0_k$, and $\widetilde{v}_1$ does not contain $\widetilde{v}_2$.

To prove (b) and (c) consider a free edge e of the triangulation T and assume that e is Delaunay-flippable. There are $z, z' \in \mathbb{Z}$ such that $e = s_{z,z'}$. The segment $\widetilde{e} = [\widetilde{p}^0_z, \widetilde{p}^1_{z'}]$ is a lift of e so it is incident to two faces $\widetilde{t_1}$ and $\widetilde{t_2}$ of the lift $\widetilde{T}$ of T. Let $\widetilde{v}_1$ be the vertex of $\widetilde{t_1}$ that is not a vertex of $\widetilde{t_2}$ and let $\widetilde{v}_2$ be the vertex of $\widetilde{t_2}$ that is not a vertex of $\widetilde{t_1}$. Up to renaming $\widetilde{v}_1$ and $\widetilde{v}_2$ there is $\epsilon \in \{1, -1\}$ such that $\widetilde{v}_1 = \widetilde{p}^0_{z-\epsilon}$ and $\widetilde{v}_2 = \widetilde{p}^1_{z'+\epsilon}$: every other case would contradict the fact that both T and the triangulation resulting from the flip of e in T are indeed triangulations. The edge e' resulting from the lift of e in T admits the segment $[\widetilde{v}_1, \widetilde{v}_2]$ as a lift and $l(e') \geq l(e) - 2/n$.

To prove (d) consider a lift $\widetilde{e}$ of a free edge e of T and the two vertices $\widetilde{v_1}$ and $\widetilde{v_2}$ of $\widetilde{e}$. Let τ_1 be the translation by the vector $(1, 0)$ (one of the two translations defining $\mathbb{T}_\square$). The four points of $\mathbb{R}^2$ that are $\widetilde{v_1}$, $\widetilde{v_2}$, $\tau_1(\widetilde{v_2})$ and $\tau_1(\widetilde{v_1})$ are the vertices of a closed parallelogram $P_\diamond$. The closed parallelogram $P_\diamond$ contains a lift of every free edge of T. Indeed every free edge f of T distinct from e admits a lift $\widetilde{f}$ whose interior intersects the interior of $P_\diamond$ [2], and the interior of $\widetilde{f}$ cannot intersect a side of $P_\diamond$ because that would imply that the interior of the edge f intersects another edge of the triangulation T. To conclude observe that by construction the sides of $P_\diamond$ are of length 1 (for the sides $\widetilde{v_1}\tau_1(\widetilde{v_1})$ and $\widetilde{v_2}\tau_1(\widetilde{v_2})$) and of length $l(e)$ (for the sides $\widetilde{v_1}\widetilde{v_2}$ and $\tau_1(\widetilde{v_1})\tau_1(\widetilde{v_2})$)). Thus every free edge of T has its length between $l(e) - 2$ and $l(e) + 2$. $\square$

[2] The closed parallelogram $P_\diamond$ is a fundamental domain for the flat torus $\mathbb{T}_\square$.

4 Upper bound

In Section 3 we exhibited a family of triangulations T for which the length of a sequence of Delaunay flips starting from T and ending at a Delaunay triangulation is bounded from below (Proposition 3). In this section we show that our construction was actually "the worst possible" and that the lower bound of Proposition 3 is asymptotically matched by a general upper bound over all possible starting triangulations on a flat torus. This upper bound comes from an observation formalized by Proposition 4. Informally, given two "long" edges e_1 and e_2 among the edges flipped in a sequence of Delaunay flips, if e_1 and e_2 have "comparable" lengths then they must be "roughly parallel".

4.1 Statement of Proposition 4

Consider a flat torus $\mathbb{T}_\Gamma$. We say that a segment s of $\mathbb{T}_\Gamma$ *follows* a segment s' of $\mathbb{T}_\Gamma$ (possibly with $s = s'$) if there are triangulations T and T' of $\mathbb{T}_\Gamma$ (possibly with $T = T'$) such that s is an edge of T, s' is an edge of T', and there is a sequence of Delaunay flips (possibly of length 0) starting from T' and ending at T.

We map every segment s of $\mathbb{T}_\Gamma$ to a pair $\{\widetilde{p}, -\widetilde{p}\}$ of opposite nonzero vectors of $\mathbb{R}^2$ as follows. We consider the endpoints $\widetilde{u}$ and $\widetilde{v}$ of a lift of s and define the point $\widetilde{p}$ as the image of $0_{\mathbb{R}^2}$ under the translation that maps $\widetilde{u}$ to $\widetilde{v}$. The resulting pair $\{\widetilde{p}, -\widetilde{p}\}$ does not depend on the choice of $\widetilde{u}$ and $\widetilde{v}$. We call these two points the *signature points* of the segment s.

Consider two segments s and s' of $\mathbb{T}_\Gamma$ and assume that s and s' have the same endpoints u and v (u and v may be equal) and the same signature points $\widetilde{p}$ and $-\widetilde{p}$. Consider also a lift $\widetilde{u}$ of u. For $\epsilon \in \{1, -1\}$ let $\widetilde{v}_\epsilon$ denote the image of $\widetilde{u}$ under the translation that maps $0_{\mathbb{R}^2}$ to $\epsilon\widetilde{p}$. There are $\epsilon, \epsilon' \in \{1, -1\}$ such that the segment $[\widetilde{u}, \widetilde{v}_\epsilon]$ of $\mathbb{R}^2$ is a lift of s and such that the segment $[\widetilde{u}, \widetilde{v}_{\epsilon'}]$ of $\mathbb{R}^2$ is a lift of s'. If $\epsilon = \epsilon'$ then $s = s'$. Thus there cannot be more than two distinct segments of $\mathbb{T}_\Gamma$ having the same endpoints and the same signature points.

Proposition 4 *Given a flat torus $\mathbb{T}_\Gamma$ there are $\kappa > 0$ and $l_0 > 0$ depending only on $\mathbb{T}_\Gamma$ such that the following holds. If a segment s of $\mathbb{T}_\Gamma$ follows a segment s' of $\mathbb{T}_\Gamma$ and if $l(s) > l_0$ and $l(s') \in [l(s)/2, 2l(s)]$ then the signature points of s' are at distance at most κ from the line containing the signature points of s.*

See Figure 5 for an illustration of Proposition 4.

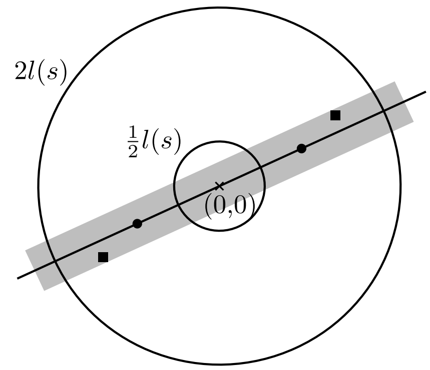

Figure 5: Illustration of Proposition 4. (Black disks) Signature points of s. (Black squares) Signature points of s'. (Gray) Points at distance at most κ from the line containing the signature points of s.

4.2 Proof of Proposition 4

Lemma 5 *Assume that a segment s of a flat torus $\mathbb{T}_\Gamma$ follows a segment s' of $\mathbb{T}_\Gamma$ and consider a lift $\widetilde{s}$ of s and a lift $\widetilde{s'}$ of s'. If $\widetilde{s}$ and $\widetilde{s'}$ intersect in their respective interiors and if there is an open disk $\widetilde{D}$ whose boundary $\partial\widetilde{D}$ contains the two endpoints of $\widetilde{s}$ and one of the endpoints of $\widetilde{s'}$ then the other endpoint of $\widetilde{s'}$ lies outside $\widetilde{D}$.*

Observe that in Lemma 5 if a point lies outside the open disk $\widetilde{D}$ it may still lie within the boundary circle $\partial\widetilde{D}$. In particular the conclusion of the lemma holds when $s = s'$ and $\widetilde{s} = \widetilde{s'}$.

Proof. Let $\widetilde{u}, \widetilde{v}$ denote the two endpoints of $\widetilde{s}$, and $\widetilde{u'}, \widetilde{v'}$ denote the two endpoints of $\widetilde{s'}$. Assume that the points $\widetilde{u}, \widetilde{v}$, and $\widetilde{u'}$ belong to the circle $\partial\widetilde{D}$. The projection $\pi(\partial\widetilde{D})$ is the intersection with $\mathbb{S}_2 \setminus P$ of a plane $\mathcal{P} \subset \mathbb{R}^3$. The plane $\mathcal{P}$ bounds two closed half-spaces whose union is $\mathbb{R}^3$ and whose intersection is $\mathcal{P}$. We will show that $\pi(\widetilde{v'})$ belongs to the half-space $\mathcal{R}$ containing the point P.

There are triangulations T and T' of $\mathbb{T}_\Gamma$ such that s is an edge of T, s' is an edge of T', and there is a sequence of Delaunay flips starting from T' and ending at T. The lift $\widetilde{T}$ of T and the lift $\widetilde{T'}$ of T' are infinite triangulations of $\mathbb{R}^2$; $\widetilde{s}$ is an edge of $\widetilde{T}$ and $\widetilde{s'}$ is an edge of $\widetilde{T'}$. Let S and S' be the standard triangular surfaces associated to $\widetilde{T}$ and $\widetilde{T'}$ respectively. Lemma 2 and the transitivity of the aboveness relation imply that S is above S' (possibly with $S = S'$). Thus any point $\widetilde{p} \in \mathbb{R}^2$ of the intersection of $\widetilde{s}$ and $\widetilde{s'}$ is such that $\pi_{S'}(\widetilde{p})$ lies on the segment of $\mathbb{R}^3$ $[P, \pi_S(\widetilde{p})]$ on the line $I_{\widetilde{p}}$. (Section 2.3). The point $\pi_S(\widetilde{p})$ is the intersection with the line $I_{\widetilde{p}}$ of an edge of S: this edge is the segment of $\mathbb{R}^3$ $[\pi(\widetilde{u}), \pi(\widetilde{v})]$. This segment is fully contained in the plane $\mathcal{P}$ since its endpoints $\pi(\widetilde{u})$ and $\pi(\widetilde{v})$ both belong to $\mathcal{P}$. In particular $\pi_S(\widetilde{p})$ belongs to $\mathcal{P}$ and $\pi_{S'}(\widetilde{p})$ belongs to the half-space $\mathcal{R}$. Since $\pi_{S'}(\widetilde{p})$ is distinct from $\pi(\widetilde{u'})$ and belongs to the segment of $\mathbb{R}^3$ $\left[\pi(\widetilde{u'}), \pi(\widetilde{v'})\right]$ and since both $\pi_{S'}(\widetilde{p})$ and $\pi(\widetilde{u'})$ belong to $\mathcal{R}$ then so does $\pi(\widetilde{v'})$. □

Lemma 6 *Let $\varepsilon > 0$ and $d > 20\varepsilon$. Let $\widetilde{u} \in \mathbb{R}\times]-\infty, 0[$ and $\widetilde{v} \in \mathbb{R}\times]0, +\infty[$ such that $\|\widetilde{u}\| < \varepsilon$ and $\|\widetilde{v} - \widetilde{u}\| < 4d$. There is a unique open disk $\widetilde{D}$ whose boundary contains $\widetilde{u}$ and the points $(d, 0)$ and $(-d, 0)$. If $\widetilde{v}$ lies outside $\widetilde{D}$ then $y_{\widetilde{v}} < 100\varepsilon$ where $y_{\widetilde{v}}$ denotes the second coordinate of $\widetilde{v}$.*

Observe that in Lemma 6 if the point $\widetilde{v}$ lies outside the open disk $\widetilde{D}$ it may, still, belong to its boundary. See Figure 6 for an illustration of Lemma 6.

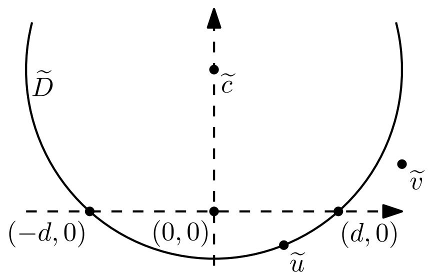

Figure 6: Illustration of Lemma 6.

We put the proof of Lemma 6 in Appendix 5. Now we prove Proposition 4 using Lemmas 5 and 6.

Proof. *(Proof of Proposition 4)*
Assume that a segment s of a flat torus $\mathbb{T}_\Gamma$ follows a segment s' of $\mathbb{T}_\Gamma$. Consider a lift $\widetilde{s}$ of s. Up to a rotation and a translation we assume that $\widetilde{s}$ is a horizontal segment whose center is the point $(0, 0)$. We claim that there exist $\varepsilon > 0$ depending only on $\mathbb{T}_\Gamma$ and a lift $\widetilde{s'}$ of s' whose endpoints $\widetilde{u} = (x_{\widetilde{u}}, y_{\widetilde{u}})$ and $\widetilde{v} = (x_{\widetilde{v}}, y_{\widetilde{v}})$ satisfy the three following conditions: $\|\widetilde{u}\| < \varepsilon$, $y_{\widetilde{u}} < 0$, and $y_{\widetilde{u}} \le y_{\widetilde{v}}$. To prove this claim start with any lift of s' and let $\widetilde{p} = (x_{\widetilde{p}}, y_{\widetilde{p}})$ and $\widetilde{q} = (x_{\widetilde{q}}, y_{\widetilde{q}})$ denote the endpoints of this lift. Up to renaming $\widetilde{p}$ and $\widetilde{q}$ we assume $y_{\widetilde{p}} \le y_{\widetilde{q}}$. There is $\varepsilon > 0$ such that any open disk of diameter ε intersects the Γ-orbit of $\widetilde{p}$. Hence there is a point $\widetilde{u} \in \mathbb{R}^2$ at distance less than $\varepsilon/2$ from the point $(0, -\varepsilon/2)$ and a translation $\tau \in \Gamma$ such that $\tau(\widetilde{p}) = \widetilde{u}$. Setting $\widetilde{v} = \tau(\widetilde{q})$ proves the claim.

The signature points of s belong to the line $\mathbb{R} \times \{0\}$. We set $\kappa = 101\varepsilon$ and consider one of the two signature points of s', namely $\widetilde{v} - \widetilde{u}$. Since $-\varepsilon < y_{\widetilde{u}} < 0$ and $y_{\widetilde{u}} \le y_{\widetilde{v}}$ proving $y_{\widetilde{v}} < 100\varepsilon$ will infer the proposition. Having $y_{\widetilde{v}} \le 0$ would conclude so we assume $y_{\widetilde{v}} > 0$. There are two cases: either $\widetilde{s}$ and $\widetilde{s'}$ intersect in their interiors or they do not.

First assume that $\widetilde{s}$ and $\widetilde{s'}$ intersect in their interiors. We set $d = l(s)/2$ and we can enforce that $d > 20\varepsilon$. Indeed we assumed $l(s) > l_0$ and we can choose l_0 large enough with respect to ε (recall that ε depends only on $\mathbb{T}_\Gamma$). Lemma 5 implies that $\widetilde{v}$ lies outside the open disk $\widetilde{D}$ whose boundary contains $\widetilde{u}$ and the endpoints $(d, 0)$ and $(-d, 0)$ of $\widetilde{s}$. Thus the conditions of Lemma 6 are satisfied and $y_{\widetilde{v}} < 100\varepsilon$.

If $\widetilde{s}$ and $\widetilde{s'}$ do not intersect in their interiors then $\widetilde{v}$ lies outside $\widetilde{D}$ and the conditions of Lemma 6 are satisfied again. □

4.3 Proof of the upper bound

Lemma 7 is folklore. We prove it in Appendix 6 for completeness.

Lemma 7 *Consider a flat torus $\mathbb{T}_\Gamma$, an integer $m \ge 0$, and a sequence of Delaunay flips $T_0, \ldots, T_m$. For every $k \in \{1, \ldots, m\}$ we let e_k denote the edge of T_{k-1} that is flipped to obtain T_k. The segments $e_1, \ldots, e_m$ of $\mathbb{T}_\Gamma$ are pairwise distinct.*

The edges flipped in a sequence of Delaunay flips are not longer than $2\Delta(T)$ where $\Delta(T)$ is a parameter measuring in some sense how "stretched" the starting triangulation T is [4, Lemma 10]. The arguments yielding a bound in terms of $\Delta(T)$ easily infer a bound in terms of the maximum length of an edge in T. This new bound is stated by Lemma 8. As the proof of Lemma 8 is only a slight adaptation of the anterior proof [4, Lemma 10] we omit it here and put it in Appendix 7 for completeness.

Lemma 8 *Consider triangulations T and T' of a flat torus $\mathbb{T}_\Gamma$ and assume that there is a sequence of Delaunay flips starting from T' and ending at T. Then the edges of T cannot be more than twice as long as a longest edge of T'.*

Now we prove Theorem 1.

Proof. *(Proof of Theorem 1)* Consider $m \ge 0$ and a sequence of Delaunay flips $T_0, \ldots, T_m$ such that $T_0 = T$. For every $k \in \{0, \ldots, m\}$ the edges of T_k constitute a set E_k of segments of $\mathbb{T}_\Gamma$. We are interested in the union E of the sets $E_0, \ldots, E_m$. By Lemma 7 the cardinal of E is not smaller than m. We partition the elements of E into $n(n+1)/2$ subsets according to their endpoints, as follows. For every unordered pair $\{u, v\}$ of vertices of the triangulation T we consider the set of segments in E that end at u and v. For every single vertex v of T we consider the set of segments in E that admit v as their unique endpoint. Proving that each of those subsets contains at most $C_\Gamma \cdot \Lambda(T)$ segments will infer the result.

So consider such a subset $F \subseteq E$ in the partition that we just described and let u and v be the (possibly equal) endpoints of the segments in F. Let $\kappa > 0$ and $l_0 > 0$ be given by Proposition 4.

As explained in Section 4.1 there cannot be more than two distinct segments of $\mathbb{T}_\Gamma$ having the same endpoints and the same signature points. Fix a lift $\widetilde{u}$ of u and a lift $\widetilde{v}$ of v. For any signature point $\widetilde{p}$ of a segment in F there is $\tau \in \Gamma$ such that either $\widetilde{p}$ or $-\widetilde{p}$ is equal to $\tau(\widetilde{v}) - \widetilde{u}$. Thus there is a finite number of such signature points that are at distance at most l_0 from the point $(0,0)$, and this finite number depends only on $\mathbb{T}_\Gamma$ (recall that l_0 depends only on $\mathbb{T}_\Gamma$). That implies that there is only a finite number of segments in F that are not longer than l_0.

Consequently we let $F' \subseteq F$ be the set of segments in F that are longer than l_0: we will now bound the cardinality of F'. By Lemma 8 no segment in F' is longer than $2\Lambda(T)$. We partition the segments in F' by their lengths as follows. We consider $j_0 = l_0 < j_1 < \cdots < j_N = 2\Lambda(T)$ for some $N \geq 1$ such that for every $k \in \{1, \ldots, N\}$ the reals j_{k-1} and j_k differ by a factor of at most 2. For every $k \in \{1, \ldots, N\}$ we let F'_k denote the set of segments in F' whose length belongs to $]j_{k-1}, j_k]$. We now fix k and claim that F'_k contains at most $C'_\Gamma \cdot (j_k - j_{k-1})$ segments, where $C'_\Gamma > 0$ is a constant that depends only on $\mathbb{T}_\Gamma$.

To prove this claim observe that if F'_k is not empty then it contains a segment s that follows every other segment $s' \in F'_k \setminus \{s\}$. For such another segment s' Proposition 4 states that the signature points of s' are at distance at most κ from the line containing the signature points of s. Also the distance to $(0,0)$ of the two signature points of s' is the length of s' and thus lies between j_{k-1} and j_k. Consequently the number of signature points of elements of F'_k is at most linear in $j_k - j_{k-1}$ and the constant coefficient depends only on $\mathbb{T}_\Gamma$.

To clarify this statement observe that the signature points of elements of F'_k all belong, by definition, to the Γ-orbit $\mathcal{O}$ of some point of $\mathbb{R}^2$. We just proved that such signature points also belong to the set $\mathcal{D}$ of points of $\mathbb{R}^2$ (1) that are at distance κ from the line containing the signature points of s and (2) whose distance to $(0,0)$ lies between j_{k-1} and j_k. The cardinality of the intersection of $\mathcal{O}$ and $\mathcal{D}$ is linear in $j_k - j_{k-1}$, and the constant coefficient depends only on $\mathcal{O}$ and κ, that both depend only on $\mathbb{T}_\Gamma$.

That, together with Proposition 3 for the lower bound, concludes the proof of Theorem 1. □

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Appendix

5 Proof of Lemma 6

Proof. We write $\widetilde{u} = (x_{\widetilde{u}}, y_{\widetilde{u}})$ and $\widetilde{v} = (x_{\widetilde{v}}, y_{\widetilde{v}})$ and recall that $y_{\widetilde{v}} > 0$ and $y_{\widetilde{u}} < 0$ both hold by assumption. The latter enforces the existence of the open disk $\widetilde{D}$. Now let $\widetilde{c}$ denote the center of $\widetilde{D}$. The segment $[-d, d] \times \{0\}$ is a chord of $\widetilde{D}$ and its midpoint is the point $(0, 0)$. Thus the first coordinate of $\widetilde{c}$ is 0 and the radius of $\widetilde{D}$ is $\sqrt{y_{\widetilde{c}}^2 + d^2}$ where $y_{\widetilde{c}}$ denotes the second coordinate of $\widetilde{c}$. One easily gets $y_{\widetilde{c}} > 0$ from the assumptions $y_{\widetilde{u}} < 0$, $\|\widetilde{u}\| < \varepsilon$, and $d > \varepsilon$. See Figure 6.

We first prove a few inequalities that may seem arbitrary at first but will be used in the end of the proof. Pythagorean Theorem gives $(y_{\widetilde{c}} - y_{\widetilde{u}})^2 + x_{\widetilde{u}}^2 = y_{\widetilde{c}}^2 + d^2$ which simplifies to $-2y_{\widetilde{u}}y_{\widetilde{c}} = d^2 - x_{\widetilde{u}}^2 - y_{\widetilde{u}}^2$. We assumed $\|\widetilde{u}\| < \varepsilon$ and $d > \sqrt{2}\varepsilon$, that implies $x_{\widetilde{u}}^2 + y_{\widetilde{u}}^2 < d^2/2$ and $-y_{\widetilde{u}} < \varepsilon$ and thus

$$4\varepsilon y_{\widetilde{c}} > d^2. \tag{1}$$

Equation (1) combined with the assumption that $d > 20\varepsilon$ implies

$$y_{\widetilde{c}} > 100\varepsilon. \tag{2}$$

The triangular inequality gives $\|\widetilde{v}\| \leq \|\widetilde{v} - \widetilde{u}\| + \|\widetilde{u}\|$. The later is smaller than $4d + \varepsilon < 5d$ by assumptions. So $\|\widetilde{v}\|^2 < 25d^2$ and by Equation 1 we obtain

$$\|\widetilde{v}\|^2 < 100\varepsilon y_{\widetilde{c}}. \tag{3}$$

Equation (3) and Equation (2) imply

$$\|\widetilde{v}\| < y_{\widetilde{c}}. \tag{4}$$

Now we prove $y_{\widetilde{v}} < 100\varepsilon$. Since $\widetilde{v}$ lies outside $\widetilde{D}$ then $(y_{\widetilde{c}} - y_{\widetilde{v}})^2 + x_{\widetilde{v}}^2 \geq y_{\widetilde{c}}^2 + d^2$, which simplifies to $y_{\widetilde{v}}^2 - 2y_{\widetilde{c}}y_{\widetilde{v}} + x_{\widetilde{v}}^2 - d^2 \geq 0$. We study this inequality to derive a bound on $y_{\widetilde{v}}$. Equation (4) implies $4(y_{\widetilde{c}}^2 + d^2 - x_{\widetilde{v}}^2) > 0$ hence the polynomial $X^2 - 2y_{\widetilde{c}}X + x_{\widetilde{v}}^2 - d^2$ univariate in X admits two real roots $y_{\widetilde{c}} \pm \sqrt{y_{\widetilde{c}}^2 + d^2 - x_{\widetilde{v}}^2}$. Equation (4) enforces $y_{\widetilde{v}} \leq y_{\widetilde{c}} - \sqrt{y_{\widetilde{c}}^2 + d^2 - x_{\widetilde{v}}^2}$, which implies

$$y_{\widetilde{v}} < y_{\widetilde{c}} \left(1 - \sqrt{1 - x_{\widetilde{v}}^2 / y_{\widetilde{c}}^2}\right).$$

Equation (3) and Equation (2) successively infer

$$y_{\widetilde{v}} < y_{\widetilde{c}} \left(1 - \sqrt{1 - 100\varepsilon / y_{\widetilde{c}}}\right) \leq 100\varepsilon.$$

That proves the lemma. □

6 Proof of Lemma 7

Proof. Assume there are $k, k' \in \{1, \ldots, m\}$ such that $k < k'$ and $e_k = e_{k'}$. Let S_{k-1}, S_k and $S_{k'-1}$ be the standard triangular surfaces associated to the lifts of T_{k-1}, T_k and $T_{k'-1}$, respectively. Consider the edge f of T_k resulting from the Delaunay flip of the edge e_k in T_{k-1}. Let $p \in \mathbb{T}_\Gamma$ be the intersection point of the interiors of f and e_k. Let also $\widetilde{p} \in \mathbb{R}^2$ be a lift of p.

Since $e_k = e_{k'}$ then $\pi_{S_{k-1}}(\widetilde{p}) = \pi_{S_{k'-1}}(\widetilde{p})$. By Lemma 2 $S_{k'-1}$ is above S_k and S_k is above S_{k-1}. We deduce $\pi_{S_k}(p) = \pi_{S_{k-1}}(\widetilde{p}) = \pi_{S_{k'-1}}(\widetilde{p})$. But Lemma 2 also gives $\pi_{S_k}(\widetilde{p}) \neq \pi_{S_{k-1}}(\widetilde{p})$ hence a contradiction. □

7 Proof of Lemma 8

Proof. Let $\Lambda(T')$ be the maximum length of an edge of T' and assume that there is an edge e of T such that $l(e) > 2\Lambda(T')$. Consider a lift $\widetilde{e}$ of e and let $\widetilde{p} \in \mathbb{R}^2$ be the middlepoint of $\widetilde{e}$. There is a face $\widetilde{t'}$ of the lift $\widetilde{T'}$ of T' such that $\widetilde{p}$ belongs either to $\widetilde{t'}$ or to the boundary of $\widetilde{t'}$. The three edges of the triangle $\widetilde{t'}$ are not longer than $\Lambda(T)$ so, by the triangular inequality, the distance from $\widetilde{p}$ to any vertex of $\widetilde{t'}$ is not greater than $\Lambda(T)$ and the *closed* disk $\widetilde{D} \subset \mathbb{R}^2$ of diameter $\Lambda(T)$ and centered at $\widetilde{p}$ contains $\widetilde{t'}$. Also the two endpoints $\widetilde{u}$ and $\widetilde{v}$ of $\widetilde{e}$ lie outside $\widetilde{D}$.

Consider the standard triangular surfaces S and S' associated to the lifts of T and T', respectively. The projection $\pi(\partial\widetilde{D})$ of the boundary $\partial\widetilde{D}$ of $\widetilde{D}$ is the intersection with $\mathbb{S}_2 \setminus P$ of a plane $\mathcal{P} \subset \mathbb{R}^3$. The plane $\mathcal{P}$ bounds two *open* half-spaces. The points $\pi(\widetilde{u})$ and $\pi(\widetilde{v})$ both belong to the half-space R that contains P. Thus $\pi_S(\widetilde{p}) \in R$. The vertices $\widetilde{w_1}, \widetilde{w_2}$ and $\widetilde{w_3}$ of $\widetilde{t'}$ all belong to $\partial\widetilde{D}$ thus $\pi(\widetilde{w_1}), \pi(\widetilde{w_2})$ and $\pi(\widetilde{w_3})$ all belong to $\mathcal{P}$ and $\pi_{S'}(\widetilde{p}) \in \mathcal{P}$. Consequently $\pi_{S'}(\widetilde{p})$ does not lie on the segment $[P, \pi_S(\widetilde{p})]$ of $\mathbb{R}^3$, contradicting Lemma 2. □